Practice Test 2 for Calculus II, Math 1502, September 29, 2010

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Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (25 points) Decide whether the following series converge or diverge: State which kind of convergence test you are using.

a)

$$\sum_{k=0}^{\infty} \frac{(3k)!k!}{[(2k)!]^2}$$

Because of the factorials, it is natural to use the ratio test: We have to compute

$$\frac{a_{k+1}}{a_k} = \frac{(3(k+1))!(k+1)!}{[(2(k+1))!]^2} \frac{[(2k)!]^2}{(3k)!k!}$$

Now

$$(3(k+1))! = (3k+3)! = (3k+3)(3k+2)(3k+1)(3k)! \ ,$$

$$(k+1)! = (k+1)k! \ ,$$

and

$$[(2(k+1))!]^2 = [(2k+2)!]^2 = [(2k+2)(2k+1)(2k)!]^2$$
$$= (2k+2)^2(2k+1)^2[(2k)!]^2 \ .$$

Thus

$$\frac{a_{k+1}}{a_k} = \frac{(3k+3)(3k+2)(3k+1)(k+1)}{(2k+2)^2(2k+1)^2} \ .$$

As $k \to \infty$ we find

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{3^3}{2^4} = \frac{27}{16} > 1 \ .$$

Thus the series is divergent.

b)

$$\sum_{k=2}^{\infty} \log\left(1 - \frac{1}{k^2}\right)$$

The ratio test as well as the root test are not conclusive in this example. However, since

$$1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k+1)(k-1)}{k^2}$$

we find that

$$\log\left(1 - \frac{1}{k^2}\right) = \log(k+1) + \log(k-1) - 2\log k$$

and maybe one can reduce the problem to telescoping sum. Thus, the N-th partial sum is given by

$$\sum_{k=2}^{N} [\log(k+1) + \log(k-1) - 2\log k] \ .$$

Now, we pull this sum apart and get

$$\sum_{k=2}^{N} \log(k+1) + \sum_{k=2}^{N} \log(k-1) - 2\sum_{k=2}^{N} \log k \ .$$

Now it is evident what is going on. Shifting the summation index the fist sum can be written as

$$\sum_{k=3}^{N+1} \log k \ ,$$

and the second

$$\sum_{k=1}^{N-1} \log k \ .$$

In total we have

$$\sum_{k=3}^{N+1} \log k + \sum_{k=1}^{N-1} \log k - 2\sum_{k=2}^{N} \log k \ .$$

Note that the summands with index between $k = 3$ up to $N - 1$ show up in all the sums and hence cancel out. So we are left with

$$\log(N+1) + \log N + \log 1 + \log 2 - 2\log 2 - 2\log N \ ,$$

which can be rewritten as

$$\log \frac{N(N+1)}{N^2} - \log 2 \ .$$

Thus the N-th partial sum is exactly this expression. As N tends to infinity, this expression converges to $-\log 2$. Thus, not only do we know that this series converges, but we also know its limit, namely $-\log 2$.

c)

$$\sum_{k=2}^{\infty} \frac{1}{k[\log k]^2}$$

Likewise, here the ratio and the root test are not conclusive. The function

$$\frac{1}{x[\log x]^2}$$

is a monotone decreasing positive function for $x > 2$ and hence we may use the integral test. The integral

$$\int_2^N \frac{1}{x[\log x]^2} dx$$

can be easily computed using the substitution $u = \log x$. The

$$du = \frac{dx}{x}$$

and hence

$$\int_2^N \frac{1}{x[\log x]^2} dx = \int_{\log 2}^{\log N} \frac{1}{u^2} du = \frac{1}{\log 2} - \frac{1}{\log N}$$

which converges as $N \to \infty$. hence the series converges.

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II: (25 points) a) Find the Taylor series for the function

$$f(x) = \int_0^x e^{-y^2}\, dy$$

Find a polynomial that approximates $f(x)$ on the interval $[0, 1]$ with an error less than 10^{-3}.

We use the Taylor series

$$e^{-z} = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{k!}$$

and setting $z = y^2$, we find

$$e^{-y^2} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{k!}$$

which is an alternating series! The term

$$\frac{y^{2k}}{k!}$$

tends to zero as $k \to \infty$ for every y. If we set

$$s_N(y) = \sum_{k=0}^{N} (-1)^k \frac{y^{2k}}{k!}$$

we have, by the general theory of alternating series that

$$\left| e^{-y^2} - s_N(y) \right| \le \frac{y^{2N+2}}{(N+1)!} \ .$$

Now

$$\left|\int_0^x e^{-y^2}dy - \int_0^x s_N(y)dy\right| \le \int_0^x \frac{y^{2N+2}}{(N+1)!}dx = \frac{x^{2N+3}}{(2N+3)(N+1)!}\ .$$

Since

$$\int_0^x s_N(y)dy = \int_0^x \left[\sum_{k=0}^{N}(-1)^k \frac{y^{2k}}{k!}\right] dx = \sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{(2k+1)k!}$$

we find that

$$\left|\int_0^x e^{-y^2}dy - \sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{(2k+1)k!}\right| \le \frac{x^{2N+3}}{(2N+3)(N+1)!}\ .$$

Since $0 \le x \le 1$ the term

$$\frac{x^{2N+3}}{(2N+3)(k+1)!} \le \frac{1}{(2N+3)(N+1)!}$$

With a little trial and error we find that when $N = 5$

$$\frac{1}{(2N+3)(N+1)!} = \frac{1}{13 \times (5+1)!} = \frac{1}{13 \cdots 720} = \frac{1}{9360}$$

which is a tad bigger than $\frac{1}{1000}$. Hence $N = 6$ will certainly do it. In fact we get that for $N = 6$

$$\frac{1}{(2N+3)(N+1)!} = \frac{1}{15 \cdot 5040} = \frac{1}{75600}\ .$$

Thus, the polynomial

$$\sum_{k=0}^{6}(-1)^k \frac{x^{2k+1}}{(2k+1)k!}$$

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!} + \frac{x^{13}}{13 \cdot 6!}$$

yields all the values of $f(x)$ for $0 \leq x \leq 1$ with an accuracy less than

$$\frac{1}{75600} \ .$$

b) Find the Taylor series of the function

$$\frac{1}{4-3x}$$

Write

$$\frac{1}{4-3x} = \frac{1}{4}\frac{1}{1-\frac{3}{4}x}$$

and use the geometric series to obtain the power series expansion

$$\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k x^k \ .$$

c) Sum the series

$$\sum_{k=1}^{\infty}(-1)^k k\left(\frac{3}{4}\right)^k$$

Differentiating the geometric series we find for $|x| < 1$ that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} \ .$$

Thus, if we replace x by $-\frac{3}{4}$ we find

$$\frac{1}{(1+\frac{3}{4})^2} = \sum_{k=1}^{\infty} k(-1)^{k-1}\left(\frac{3}{4}\right)^{k-1}$$

which is almost what we want. All we have to do I to multiply this result with $(-1)\frac{3}{4}$ and we obtain

$$\sum_{k=1}^{\infty}(-1)^k k\left(\frac{3}{4}\right)^k = -\frac{3}{4}\frac{1}{(1+\frac{3}{4})^2} = -\frac{12}{49} \ .$$

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III: (25 points) a) Find the interval of convergence of the power series

a)

$$\sum_{k=1}^{\infty} \frac{1}{k}(x-2)^k 2^{-k}$$

Using the ratio test we find with

$$a_k = \frac{1}{k}\frac{1}{2^k}|x-2|^k$$

that

$$\frac{a_{k+1}}{a_k} = \frac{k|x-2|}{2(k+1)}$$

which converges to $\frac{|x-2|}{2}$ as $k \to \infty$, and this limit has to be strictly less than 1, should this series converge. Hence the interval of convergence contains the interval $(0,4)$. it remains the check the endpoints. At $x=4$ the series is the harmonic series which diverges. At $x=0$ the series is the alternating harmonic series, which converges. Thus, the interval of convergence is $[0,4)$.

b)

$$\sum_{k=1}^{\infty} \frac{[\log(k)]^k}{k!} x^k$$

This is a bit tricky. The numerator calls for the root test and the denominator for the ratio test. We try the ratio test because it is harder to understand k-th roots of $k!$ as $k \to \infty$. Thus, the ratio we have to study is

$$\frac{(\log(k+1))^{k+1}}{(k+1)(\log k)^k}|x| = \frac{\log(k+1)}{(k+1)}\left(\frac{\log(k+1)}{\log k}\right)^k |x|$$

The tricky term is

$$\left(\frac{\log(k+1)}{\log k}\right)^k = \left(\frac{\log k + \log(1+\frac{1}{k})}{\log k}\right)^k = \left(1 + \frac{\log(1+\frac{1}{k})}{\log k}\right)^k .$$

$$\log(1+\frac{1}{k}) = \int_1^{1+\frac{1}{k}} \frac{1}{x} dx \leq 1 \times \int_1^{1+\frac{1}{k}} dx = \frac{1}{k} \ .$$

Thus

$$\left(1+\frac{\log(1+\frac{1}{k})}{\log k}\right)^k \leq \left(1+\frac{1}{k\log k}\right)^k < (1+\frac{1}{k})^k$$

for k sufficiently large. Thus, whatever the expression on the left of the above expression converges to, it must be less than $e < 3$. The remaining factor

$$\frac{|x|\log(k+1)}{k+1}$$

tends to zero as $k \to \infty$ for every value of x. Hence the interval of convergence is the whole real line.

c)

$$\sum_{k=2}^{\infty} \frac{\log k}{k^2} x^k$$

This example is straightforward. Applying the ratio test we have to calculate

$$\lim_{k\to\infty} \frac{\log(k+1)K^2}{(k+1)^2 \log k}|x| = |x| \ .$$

Hence the interval of convergence contains $(-1, 1)$. Next, we consider the endpoints. At $x + 1$ the series has the form

$$\sum_{k=2}^{\infty} \frac{\log k}{k^2}$$

which converges by comparing $\frac{\log k}{k^2}$ with $\frac{1}{k^{3/2}}$ using the p-test and the comparison test. Since the series converges absolutely, we find that it also converges at $x = -1$ and hence the interval of convergebce is $[-1, 1]$.

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IV: (25 points) Solve the initial value problems

a)

$$y'' - 2y' + 5y = 0 \ , \quad y(0) = 0 \ , \ y'(0) = 1.$$

The characteristic equation is

$$r^2 - 2r + 5 = 0 \ .$$

The roots are

$$r_1 = 1 + 2i \ , \ r_2 = 1 - 2i \ .$$

The solutions re

$$e^x \cos 2x \ , \ e^x \sin 2x \ .$$

The general solution is

$$y(x) = e^x (c_1 \cos 2x + c_2 \sin 2x) \ .$$

Since $y(0) = 0$, $c_1 = 0$. Since

$$y'(x) = e^x c_2 \sin 2x + 2e^x c_2 \cos 2x$$

$$1 = y'(0) = 2c_2$$

it follows that $c_2 = 1/2$. Thus,

$$y(x) = \frac{1}{2} e^x \sin 2x \ .$$

b)

$$y' = x(1 + y^2) \ , \ y(\frac{\pi}{2}) = 0$$

Separating variables yields

$$\frac{y'}{1 + y^2} = x$$

Integrating both sides yields

$$\tan^{-1} y = \frac{x^2}{2} + C$$

or

$$y(x) = \tan(C + \frac{x^2}{2}) \ .$$

We know that tan(0 = 0 and hence of we choose

$$C = -\frac{\pi^2}{8}$$

we have that

$$y(x) = \tan(\frac{x^2}{2} - \frac{\pi^2}{8})$$

is the right solution.

c)

$$y' + 3xy = x \ , \ y(0) = 1$$

The integrating factor is

$$e^{3x^2/2}$$

since

$$\frac{d}{dx} = 3xe^{3x^2/2} \ .$$

Multiplying the equation by $e^{3x^2/2}$ yields

$$e^{3x^2/2}y' + 3xe^{3x^2/2}y = xe^{3x^2/2}$$

or

$$(e^{3x^2/2}y)' = xe^{3x^2/2}$$

integrating both sides yields

$$e^{3x^2/2}y = \frac{1}{3}e^{3x^2/2} + C$$

and hence

$$y(x) = \frac{1}{3} + Ce^{-3x^2/2}$$

is the general solution. $y(0) = 1$ requires that

$$1 = \frac{1}{3} + C$$

or that $C = 2/3$. Hence

$$y(x) = \frac{1}{3} + \frac{2}{3}e^{-3x^2/2} \ .$$