Math 1502, **Exam 1**, September 26, 2012

SOLUTIONS

1. Consider the following linear differential equation

$$y'' - 8y' + 16y = 0$$

(a) Find the general solution $y(x)$.

Solve characteristic equation: $r^2 - 8r + 16 = 0$. The only root is $r = 4$, hence, the general solution is

$$y(x) = (ax + b)e^{4x},$$

where a, b are constants.

(b) Solve the initial value problem with the conditions $y(0) = 1$, $y'(0) = 6$.

The conditions applied to the general solutions give:

$$y(0) = b = 1$$

and

$$y'(0) = \left[ae^{4x} + 4(ax+b)e^{4x}\right]_{x=0} = a + 4b = 6.$$

Solving these two equations we get $b = 1$ and $a = 2$.

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2. Determine if the integral converges and, if so, calculate the integral whenever possible.

(a)

$$\int_0^1 \frac{x}{\sqrt{1-x}}\, dx$$

Integrating by parts, we get

$$\begin{aligned}\int_0^b \frac{x}{\sqrt{1-x}}\, dx &= \left[-2x\sqrt{1-x}\right]_0^b + 2\int_0^b \sqrt{1-x}\, dx \\ &= -2b\sqrt{1-b} + 2\left[-\frac{2}{3}(1-x)^{3/2}\right]_0^b \\ &= -2b\sqrt{1-b} - \frac{4}{3}(1-b)^{3/2} + \frac{4}{3}.\end{aligned}$$

As $b \to 1$ the above expression converges to $\frac{4}{3}$.

(b)

$$\int_0^\infty x\sin(x^2)\, dx$$

Substituting $u = x^2$, we get

$$\int_0^b x\sin(x^2)\, dx = \int_0^{\sqrt{b}} \sin u\, du = [\cos u]_0^{\sqrt{b}} = \cos(\sqrt{b}) - 1.$$

The limit of the above expression as $b \to \infty$ does not exist, hence, the integral does not converge.

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3. Does the series converge absolutely, conditionally, or diverge? Justify your answer.

(a)

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

Use ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \xrightarrow[k\to\infty]{} \frac{1}{4}$$

Since the series is positive and the ratio limit is < 1, the series converges absolutely.

(b)

$$\sum_{k=1}^{\infty} (-1)^k \ln(1+\frac{1}{k})$$

The series converges, since it is alternating and the absolute value of the term decreases and approaches 0. however, the series

$$\sum_{k=1}^{\infty} \ln(1+\frac{1}{k})$$

have the partial sum

$$s_n = \sum_{k=1}^{n} \ln(1+\frac{1}{k}) = \sum_{k=1}^{n} (\ln(k+1) - \ln k) = \ln(n+1),$$

which approaches ∞.

Therefore, the series converges conditionally.

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4. (a) Find the Taylor polynomial $P_6(x)$ of degree 6 for $f(x) = x\cos x$.

Use Taylor series for $\cos x$: since

$$P_\infty(x) = x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots),$$

the Taylor polynomial of degree 6 is

$$P_6(x) = x - \frac{x^3}{2} + \frac{x^5}{24}.$$

(b) What is radius of convergence of the Taylor series of $f(x)$ centered at $x = 0$? Justify your answer.

The radius of convergence is ∞, since the series for $\cos x$ converge for every x.

(c) Find an approximation of $f(0.1)$ with error not exceeding 0.02 by using the Taylor polynomial $P_n(x)$ with n as small as possible. Use Lagrange formula to bound the error.

The remainder can be estimated as follows:

$$|R_n| \leq \frac{M_{n+1}(0.1)^{n+1}}{(n+1)!},$$

where M_{n+1} is an upper bound on $|f^{(n+1)}|$ on the interval $(0, 0.1)$. The first two derivatives are

$$\begin{aligned} f'(x) &= \cos x - x\sin x, \\ f''(x) &= -2\sin x - x\cos x. \end{aligned}$$

In particular, $|f''(x)| \leq 3$. Therefore,

$$|R_1| \leq \frac{3 \cdot (0.1)^2}{2!} = 0.015 < 0.02$$

and $P_1(0.1) = 0.1$ falls within the specified error bound.

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5. (bonus problem, 10 points) Find the Taylor series about $a = -2$ for $f(x)$ that equals the following antiderivative

$$\int \sqrt{x+2}\,\sin(\sqrt{x+2})\,dx$$

on the domain of definition of $\sqrt{x+2}$. (Write your answer in "sigma" notation.)

Start with the series

$$y\sin y = y^2 - \frac{y^4}{3!} + \frac{y^6}{5!} - \cdots$$

Substitute $y = \sqrt{x+2}$, then

$$\sqrt{x+2}\,\sin(\sqrt{x+2}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x+2)^k}{(2k-1)!}$$

Note that

Finally, by integrating term-by-term we get

$$\int \sqrt{x+2}\,\sin(\sqrt{x+2})\,dx = C + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}(x+2)^{k+1}}{(k+1)(2k-1)!} = C + \sum_{k=2}^{\infty}\frac{(-1)^k(x+2)^k}{k(2k-3)!},$$

where C is a constant.