Practice Problem for Midterm #3, Math 1502

1. Find and simplify the following determinants:

a) $\begin{vmatrix} -1 & 0 & 5 \\ 1 & -2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$ b) $\begin{vmatrix} -1 & b \\ x & y \end{vmatrix}$ c) $\begin{vmatrix} a & 1 & 2 \\ a & 2 & -1 \\ a & -1 & 2 \end{vmatrix}$

d) $\begin{vmatrix} -1-\lambda & 0 & 5 \\ 1 & -2-\lambda & -1 \\ 3 & -1 & 2-\lambda \end{vmatrix}$ e) $\begin{vmatrix} -1 & 3 & 3 & 4 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

2. Find the characteristic polynomial, eigenvalues and eigenvectors of the following matrices:

a) $\begin{pmatrix} -2 & 4 \\ 9 & -7 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ 12 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

d) $\begin{pmatrix} -1 & 3 & 3 & 4 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 5 & -1 & 2 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$

3. Find the inverses of the following matrices, if possible:

a) $\begin{pmatrix} 1 & 3 \\ 2 & -6 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 5 \\ 3 & 2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -3 & 0 & -2 \end{pmatrix}$

4. Find the basis of the kernel and the column space of the following matrices and determine their rank.

a) $\begin{pmatrix} 1 & 3 & 2 & -3 & 1 \\ 2 & 6 & 1 & -2 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & -4 & 3 & -1 \\ -2 & 3 & -5 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 2 & 1 \\ -5 & 6 \end{pmatrix}$

5. Find the dimension and a basis of the following subspaces:

a) The span of $\begin{pmatrix} 1 \\ 6 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix}$.

b) All vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ in $\mathbb{R}^4$ with $x_1 + x_2 + x_3 + x_4 = 0$. (Hint: Think of this as a kernel of some matrix A).

c) All vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathbb{R}^3$ with $x_1 + x_2 - 3x_3 = 0$ and $x_1 - x_2 + 2x_3 = 0$ (Hint: Think of this as a kernel of some matrix A).

6. Diagonalize the following matrices, if possible:

a) $\begin{pmatrix} 1 & -2 \\ 2 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 4 & 1 \\ 0 & -4 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

7. True or False. No partial credit.

(a) A singular matrix always has determinant equal to zero.

(b) Any 4×4 matrix has 4 distinct eignevalues.

(c) Eigenvectors of any 6×6 matrix always span $\mathbb{R}^6$.

(d) A 2×6 matrix must have kernel of dimension at least 4.

(e) If a 5×7 matrix has kernel of dimension 2 then its column space is $\mathbb{R}^5$.

(f) Any invertible matrix has kernel of dimension 0.

(g) For any invertible matrix A, the determinant of A^{-1} is equal to $\frac{1}{\text{Det}\, A}$.

(h) The difference of any two vectors in a vector subspace is also in the vector subspace.

(i) If $\mathbf{u}$ is an eigenvector of A then $-\mathbf{u}$ is an eigenvector of A.

(j) If λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of A.

(k) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A corresponding to the same eigenvalue λ, then $\mathbf{u} + \mathbf{v}$ is also an eigenvector of A.

(l) If $\mathbf{u}$ and $\mathbf{v}$ are a basis of 2 dimensional subspace V, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{v}$ are also a basis of V.

(m) Any basis of a subspace must have the same number of vectors in it.

(n) If $\mathbf{u}$ is an eigenvector of A then $\mathbf{u}$ is also an eigenvector of A^{-1}.

(o) If A is diagonalizable then A^{-1} is also diagonalizable.

(p) If rank of A is 1 then all of its columns are multiples of each other.

(q) Any matrix with distinct eigenvalues is diagonalizable.

8. Find all numbers h so that the following matrix is singular:

a) $\begin{pmatrix} h & 0 & 2 \\ 1 & h & 1 \\ 1 & 3 & 1 \end{pmatrix}$ b) $\begin{pmatrix} h & h \\ 1 & h^2 \end{pmatrix}$ c) $\begin{pmatrix} h+5 & 1 & -1 \\ h+4 & 0 & 1 \\ h-3 & 3 & 1 \end{pmatrix}$

9. Construct a matrix with the following properties:

a) 2×2 matrix with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$, and corresponding eigenvectors $\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{u_2} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

b) 3×3 matrix with eigenvalues $\lambda_1 = -2$, $\lambda_2 = 0$ and $\lambda_3 = 1$, and corresponding eigenvectors $\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

10. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be a basis of $\mathbb{R}^n$. Explain why the vectors $T(\mathbf{x_1}), T(\mathbf{x_2}), \ldots, T(\mathbf{x_n})$ determine $T(\mathbf{v})$ for any $\mathbf{v}$ in $\mathbb{R}^n$. If the vectors $T(\mathbf{x_1}), T(\mathbf{x_2}), \ldots, T(\mathbf{x_n})$ span $\mathbb{R}^n$, explain why the matrix of T is invertible.

11. a) Let A and B two $n \times n$ matrices. Suppose that AB is singular. Explain why either A or B must be singular.

b) Suppose that A is an $n \times n$ matrix such that $A^2 = I$. Explain why all eigenvalues of A are either 1 or -1. (Hint: look at what happens if $A\mathbf{x} = \lambda\mathbf{x}$ with λ not equal to 1 or -1).

c) Suppose that A is an $n \times n$ matrix with columns $\mathbf{v_1}, \ldots, \mathbf{v_n}$. Let $\mathbf{e_i}$ be the vector in $\mathbb{R}^n$ with 1 in i-th entry, and 0 in all other entries. What is $A\mathbf{e_i}$? Find $A^{-1}\mathbf{v_i}$.

d) Explain why all eigenvectors of a matrix A that correspond to the same eigenvalue, form a vector subspace.