Math 1502, **Practice Problem Set for Exam 3**

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1. Find the inverses of the following matrices if possible:

   (a) $\begin{bmatrix} 2 & a \\ 0 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 3 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} x & x & x & x \\ x & 2x & x & x \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2y & y \end{bmatrix}$

2. Find the determinants of the following matrices. Which matrices are singular?

   (a) $\begin{bmatrix} x & 1 \\ y & -z \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

   (d) $\begin{bmatrix} x & x & x & x \\ 1 & 2 & 0 & 5 \\ 0 & 0 & 2y & y \\ 0 & 0 & 4 & 5 \end{bmatrix}$

3. Find

   - a basis of the null space
   - a basis of the column space
   - the rank

   of the following matrices:

   (a) $\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 2 & 8 & 12 & 7 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & 2 & 3 \\ 2 & 4 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

4. Find a basis and the dimension for the following subspaces:

   (a) Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

   (b) vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the relation $x + 2y + 3z = 0$

   (c) vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the relation $x = -2y - 3z$ and $4x + 5y + 6z = 0$

5. Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ are bases of $\mathbb{R}^2$.

   (a) Find the coordinates $[\boldsymbol{x}]_{\mathcal{B}}$ and $[\boldsymbol{x}]_{\mathcal{C}}$ (relative to the bases $\mathcal{B}$ and $\mathcal{C}$, respectively) for the vector $\boldsymbol{x} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$.

   (b) Find the change-of-coordinates matrix for the coordinate change from basis $\mathcal{B}$ to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

   (c) Find the change-of-coordinates matrix for the coordinate change from basis $\mathcal{C}$ to the standard basis.

   (d) Find the change-of-coordinates matrix for the coordinate change from the standard basis to basis $\mathcal{C}$.

   (e) Find the change-of-coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ for the coordinate change from basis $\mathcal{B}$ to basis $\mathcal{C}$.

6. Compute eigenvectors, corresponding eigenvectors (eigenspaces), and diagonalize the following matrices if possible.

   (a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$

   (c) $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 5 \end{bmatrix}$

7. (a) Show that the vectors $\boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ are orthogonal.

   (b) Find the projection of the vector $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto the subspace $L = \operatorname{Span}\{\boldsymbol{x}, \boldsymbol{y}\}$.

   (c) Find the length of the projection of $\boldsymbol{z}$ on the orthogonal complement $L^{\perp}$.

8. True or false?

   (a) For two square matrices A and B it is always true that $AB = BA$.

   (b) A 3×5 matrix must have the nullspace of dimension at least 2.

   (c) $\text{Nul}(A) = \text{Col}(A^T)$.

   (d) A regular matrix always has determinant equal to zero.

   (e) $\det A = \det A^T$.

   (f) If rank of A is 1 then all of its rows are multiples of each other.

   (g) The sum of any two vectors in a vector subspace is also in the vector subspace.

   (h) If $\{\boldsymbol{u}, \boldsymbol{v}\}$ is a basis of a subspace V , then $\{2\boldsymbol{u} + 3\boldsymbol{v}, \boldsymbol{v}\}$ is also a basis of V.

   (i) For any invertible matrix A, $\det A^{-1} = (\det A)^{-1}$.

   (j) Any $n \times n$ matrix has n distinct eigenvalues.

   (k) Eigenvectors of any $n \times n$ matrix always span $\mathbb{R}^n$.

   (l) If $\boldsymbol{x}$ is an eigenvector of A then $-2\boldsymbol{x}$ is an eigenvector of $-A$.

   (m) If 2 is an eigenvalue of A then 6 is an eigenvector of $3A$.

   (n) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors of A corresponding to the same eigenvalue, then $2\boldsymbol{u} + 3\boldsymbol{v}$ is also an eigenvector of A.

   (o) If A is diagonalizable then A^2 is also diagonalizable.

   (p) If $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{z} = 0$, then $\boldsymbol{y} \cdot \boldsymbol{z} = 0$.

   (q) If $\boldsymbol{x} \cdot \boldsymbol{y} = 0$, then the projection of $\boldsymbol{y}$ onto $\boldsymbol{x}$ is the zero vector.

   (r) The orthogonal complement to the span of 2 linearly independent vectors in $\mathbb{R}^4$ is 2-dimensional.