Solutions

Math 1512, **First Exam**, 28 Sept 2010

NAME: **Section:**

Neither cheatsheets nor calculators are allowed.

page	1	2	3	4	5	6	total
points							
maximum	20	20	20	20	20	15	

For definitions consult the textbook

1. Definitions:

   (a) Let S be a subset of a linear space V. Define what it means for S to be *a basis* of V.

   (b) Give a definition of the *dimension* of a linear space V.

   (c) Suppose V is a Euclidean space with an inner product.

      (a) Define the norm on V.

      (b) What does it mean for a basis S of V to be *orthonormal*?

2. Determine (prove/disprove) whether or not S is a subspace of the corresponding linear space in the following examples:

(a) $S = \{(x, y, z) \in \mathbb{R}^3 : x + y = 1\} \subset \mathbb{R}^3$

Let $A = (1,0,0) \in S$.

$2 \cdot A = (2,0,0) \notin S$, therefore S is not closed under the scalar multiplication

Answer: not a subspace

(b) $S = \{f \in P_n : f(0) = f(2)\} \subset P_n$

Let $f, g \in S$. Then

(1) for all $c \in \mathbb{R}$, $cf(0) = cf(2) \Rightarrow cf \in S$

(2) $(f+g)(0) = f(0) + g(0) = f(2) + g(2) = (f+g)(2)$

$\Rightarrow f+g \in S$.

Answer: S is a subspace

(c) $S = \{f \in P_n : f''(0) + f'(0) - f(0) = 0\} \subset P_n$

Let $f, g \in S$. Then

(1) for all $c \in \mathbb{R}$, $cf''(0) + cf'(0) - cf(0) = 0$

$\Rightarrow cf \in S$

(2) $(f+g)''(0) + (f+g)'(0) - (f+g)(0) =$

$= f''(0) + g''(0) + f'(0) + g'(0) - f(0) - g(0) = 0$

Answer: S is a subspace.

(P_n denotes the linear space of all real polynomials of degree at most n.)

3. Let $L(P; A) = \{P + sA\}$ and $L(Q; B) = \{Q + tB\}$ be two lines in $\mathbb{R}^3$ with $A = (1, 1, 1)$, $B = (1, 1, 3)$, $P = (4, 0, 1)$, and $Q = (x, 0, 0)$. It is known that the lines intersect.

   (a) Find x.

$$\begin{bmatrix} 4+s \\ s \\ 1+s \end{bmatrix} = \begin{bmatrix} x+t \\ t \\ 3t \end{bmatrix} \Rightarrow \begin{cases} s = t = \frac{1}{2} \\ x = 4 \end{cases}$$

   (b) Find the point of intersection.

$$\begin{bmatrix} \frac{9}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}. \quad \left(\text{Plug in } s = \tfrac{1}{2} \text{ in } \begin{bmatrix} 4+s \\ s \\ 1+s \end{bmatrix}\right)$$

   (c) Find a unit vector orthogonal to the plane containing the given lines.

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} k =$$

$$= 2i - 2j$$

$$\vec{u} = \frac{A \times B}{\|A \times B\|} = \frac{1}{\sqrt{8}}(2, -2, 0) = \frac{1}{\sqrt{2}}(1, -1, 0)$$

4. Consider the Euclidean space $C(0,1)$ of continuous functions on the interval $[0,1] \subset \mathbb{R}$ equipped with the inner product:

$$(f,g) = \int_0^1 t\, f(t)\, g(t)\, dt$$

(a) Find $\cos\alpha$ of the angle α between $f(x) = x$ and $g(x) = x^2$.

$$(f,g) = \int_0^1 t \cdot t \cdot t^2 dt = \frac{1}{5}$$

$$(f,f) = \int_0^1 t^3 dt = \frac{1}{4}$$

$$(g,g) = \int_0^1 t^5 dt = \frac{1}{6}$$

$$\cos \alpha = \frac{(f,g)}{\sqrt{(f,f)}\sqrt{(g,g)}} = \frac{\frac{1}{5}}{\frac{1}{2} \cdot \frac{1}{\sqrt{6}}} = \frac{2\sqrt{6}}{5}$$

(b) Prove that

$$\int_0^1 t\sqrt{\sin t}\sqrt{\cos t}\, dt < \sqrt{\int_0^1 t \sin t\, dt}\, \sqrt{\int_0^1 t \cos t\, dt}$$

By Cauchy-Schwarz:

$$|(f,g)| < \|f\|\,\|g\| \quad \text{if} \quad f \nparallel g$$

Let $f(x) = \sqrt{\sin x}$, $g(x) = \sqrt{\cos x}$;

both functions are nonnegative on $[0,1]$

(note: $1 < \frac{\pi}{2}$). Therefore, $(f,g) \geq 0 \Rightarrow |(f,g)| = (f,g)$

If $f \parallel g$, then $f = cg$ for some $c \in \mathbb{R}$.

But then $f(0) = c\, g(0) \Rightarrow c = 0. \Rightarrow \lightning$.

Conclusion: $(f,g) < \|f\|\,\|g\|$.

5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1,0,0,0)$, $x_2 = (-1,1,2,0)$, $x_3 = (1,2,3,1)$.

(a) Use the Gram-Schmidt procedure to obtain an orthogonal basis $S = \{y_1, y_2, y_3\}$ for V.

$$y_1 = x_1 = (1,0,0,0)$$

$$y_2 = x_2 - \frac{(x_2,y_1)}{(y_1,y_1)} y_1 = (0,1,2,0)$$

$$\tilde{y}_3 = x_3 - \frac{(x_3,y_1)}{(y_1,y_1)} y_1 - \frac{(x_3,y_2)}{(y_2,y_2)} y_2$$

$$\left\{ \begin{array}{l} (x_2,y_1) = -1 \\ (y_1,y_1) = 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} (x_3,y_1) = 1 \\ (x_3,y_2) = 8 \\ (y_2,y_2) = 5 \end{array} \right.$$

$$= \left(0, 2 - \frac{8}{5}, 3 - 2\cdot\frac{8}{5}, 1\right) = \left(0, \frac{2}{5}, -\frac{1}{5}, 1\right)$$

$$y_3 = (0,2,-1,5) = 5\tilde{y}_3$$

$$\left(\text{Check: } (y_1,y_2) = 0, \quad (y_3,y_1) = 0, \quad (y_2,y_2) = 0\right)$$

(b) Normalize elements of S to get an orthonormal basis S' for V.

$$S' = \{\underset{\underset{u_1}{\parallel}}{(1,0,0,0)}, \underset{\underset{u_2}{\parallel}}{\frac{1}{\sqrt{5}}(0,1,2,0)}, \underset{\underset{u_3}{\parallel}}{\frac{1}{\sqrt{30}}(0,2,-1,5)}\}$$

(c) Find the coordinates (= components) of the vector $v = (0,3,5,1)$ with respect to the basis S'.

$$(v,u_1) = 0$$

$$(v,u_2) = \frac{13}{\sqrt{5}}$$

$$(v,u_3) = \frac{6}{\sqrt{30}}$$

Answer: $v = \underline{\underline{0}}\, u_1 + \underline{\underline{\frac{13}{\sqrt{5}}}}\, u_2 + \underline{\underline{\frac{6}{\sqrt{30}}}}\, u_3$.

6. Use identities involving the dot- and cross-product in the vector space $\mathbb{R}^{\overset{3}{2}}$ to prove Heron's formula for the area S of a triangle in terms of the lengths of its sides a, b, c:

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = \frac{a+b+c}{2}$ is the semi-perimeter of the triangle.
(Hint: Assume the vertices of the triangle are $0, A, B \in \mathbb{R}^{\overset{3}{\cancel{2}}}$. Let $\|A\| = a$, $\|B\| = b$, $\|A - B\| = c$.)

$2S = \|A \times B\|$, since the geometric meaning of $\|A\times B\|$ is the area of the parallelogram based on A and B.

$$\|A\times B\|^2 = \|A\|^2\|B\|^2 - (A\cdot B)^2 = a^2b^2 - (A\cdot B)^2 \qquad (*)$$

Note that $\|A - B\|^2 = \|A\|^2 + \|B\|^2 + 2A\cdot B$

$$\Rightarrow A\cdot B = \frac{1}{2}(c^2 - a^2 - b^2) \qquad (**)$$

$$(*) \text{ and } (**) \Rightarrow 4S^2 = a^2b^2 - \frac{1}{4}(c^2-a^2-b^2)^2 =$$

$$= \left(ab - \frac{1}{2}(c^2-a^2-b^2)\right)\left(ab + \frac{1}{2}(c^2-a^2-b^2)\right)$$

$$= \frac{1}{4}\left(-c^2 + (a+b)^2\right)\left(c^2 - (a-b)^2\right)$$

$$= \frac{1}{4}(a+b-c)(a+b+c)(c-a+b)(c+a-b)$$

$$\Leftrightarrow S^2 = \underset{\underset{\frac{p}{2}}{\parallel}}{\frac{a+b+c}{2}} \cdot \underset{\underset{\frac{p-c}{2}}{\parallel}}{\frac{a+b-c}{2}} \cdot \underset{\underset{\frac{p-b}{2}}{\parallel}}{\frac{a-b+c}{2}} \cdot \underset{\underset{\frac{p-a}{2}}{\parallel}}{\frac{-a+b+c}{2}} .$$