Math 1512, **Second Exam**, November 2, 2010

NAME: SOLUTIONS **Section:**

Neither cheat-sheets nor calculators are allowed.

page	1	2	3	4	5	6	total
points							
maximum	20	20	20	20	20	15	

1. Let $T : V \to W$ be a linear map.

   (a) Define the image (a.k.a. range)

   $T(V) = \{Y \in V \mid T(X) = Y \text{ for some } X \in V\}$

   (b) Define the kernel (a.k.a. nullspace)

   $N(T) = \{X \in V \mid T(X) = 0\}$

   (c) Suppose $V = W$ and $N(T) = T(V)$. What are the kernel and image of $T^2 = T \circ T$?

   $N(T^2) = V$, $T^2(V) = 0$

   (d) Let $V = W = P_2$, the space of polynomials of degree at most 2, with the ordered basis $B = \{1, x, x^2\}$. Let $T = D^2 + D$ where D is the differentiation operator. Compute matrix A corresponding to the linear map T in the basis B.

   $T(1) = 0 \leftrightarrow (0,0,0)$

   $T(x) = 1 \leftrightarrow (1,0,0)$

   $T(x^2) = 2 + 2x \leftrightarrow (2,2,0)$

   $$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   The element $a + bx + cx^2 \leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ maps to $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b + 2c \\ 2c \\ 0 \end{bmatrix} \leftrightarrow (b+2c) + 2cx$

2. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & a & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & a & b & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & a & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & a-1 & b-a & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & a-1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & b-a & 1 & -1 & -(a-1) & a-1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -(a-1) \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -(a-1) & a-1-(b-a) \\ & \cdots & & & & \cdots & & \\ & & & & & & & \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1-a & 2a-b-1 \\ 0 & 1 & -1 & 1-a \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. (a) Find the dimension and a basis for the subspace of $\mathbb{R}^5$ consisting of all solutions to the following system of homogeneous linear equations:

$$x + y + 2z + 2u + 2v = 0$$
$$x + y + 2z + 3u + 4v = 0$$
$$3x + 3y + 6z + 7u + 8v = 0$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 4 \\ 3 & 3 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free: $\underbrace{x_2 \; x_3 \quad x_5}_{\dim = 3}$

$x_1 = -x_2 - 2x_3 + 2x_5$

$x_4 = -2x_5$

Basis: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(b) Find the general solution of

$$x + y + 2z + 2u + 2v = 1$$
$$x + y + 2z + 3u + 4v = 2$$
$$3x + 3y + 6z + 7u + 8v = 4$$

Reduce: $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_2 = x_3 = x_5 = 0$, then $x_1 = -1$, $x_4 = 1$

$$X = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

4. Consider the matrix

$$A = \begin{pmatrix} 1 & x & 0 & 0 \\ x & 1 & x & 0 \\ 0 & x & 1 & x \\ 0 & 0 & x & 1 \end{pmatrix}$$

(a) Compute $\det A$.

$$\det A = 1 \cdot \begin{vmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{vmatrix} - x \cdot \begin{vmatrix} x & x & 0 \\ 0 & 1 & x \\ 0 & x & 1 \end{vmatrix} =$$

$$= 1 \cdot \begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} - x \begin{vmatrix} x & x \\ 0 & 1 \end{vmatrix} - x^2 \begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} = -x^2+1-x^2-x^2(-x^2+1)$$

$$= x^4 - 3x^2 + 1$$

(b) For what values of x is matrix A regular (a.k.a. nonsingular/invertible)?

A regular $\Leftrightarrow$ $\det A \neq 0$

$y = x^2$ $\leftarrow$ substitution

$y^2 - 3y + 1 = 0 \Rightarrow y = \dfrac{3 \pm \sqrt{5}}{2}$

<u>Answer:</u>

$x \neq \pm \sqrt{\dfrac{3 \pm \sqrt{5}}{2}}$ $\left(\text{Could be simplified: e.g. } \dfrac{6 + 2\sqrt{5}}{4} = \dfrac{(1+\sqrt{5})^2}{4} \right)$

or $x \neq \pm \dfrac{1 \pm \sqrt{5}}{2}$

5. (a) Find all (a, b) such that the system of linear equations

$$\begin{aligned} x + 2y &= a \\ 2x - 3y &= b \\ x + y &= 0 \end{aligned}$$

has a solution.

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 2 & -3 & b \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -7 & b-2a \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & a \\ 0 & 0 & b-2a+7a \end{array}\right]$$

The system has a solution as long as

$b + 5a = 0$.

(b) Find the least squares solution of the system of equations

$$\begin{aligned} x + 2y &= 1 \\ 2x - 3y &= 0 \\ x + y &= 0 \end{aligned}$$

$A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A^TA = \begin{bmatrix} 6 & -3 \\ -3 & 14 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -3 & 14 & 2 \\ 0 & 25 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & -1 & -1 \\ 0 & 5 & 1 \end{array}\right] \Leftrightarrow \begin{array}{l} +3x + y = 1 \\ y = \frac{1}{5} \end{array}$$

Answer: $y = \frac{1}{5}$, $x = \frac{4}{15}$.

6. (Bonus question: 15 points) Consider an *exact sequence* of linear maps

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-2} \xrightarrow{T_{n-1}} V_n$$

i.e., the maps $T_i : V_i \to V_{i+1}$ are such that $T_i(V_i) = N(T_{i+1})$ for $i = 1, \ldots, n-1$.
Assuming that all V_i are finite dimensional linear spaces, prove that

$$\sum_{i=1}^{n-1} (-1)^{\overset{i+1}{\cancel{i}}} \dim V_i = \dim(N(T_1)) + (-1)^n \dim(T_{n-1}(V_{n-1})).$$

Proof. Induction on n.

For $n=2$ $\quad \dim V_1 = \dim(N(T_1)) + \dim(T_1(V_1))$ holds by the "rank + nullity" thm.

Assume the formula holds for $n \leftarrow n-1$:

$$(*) \quad \sum_{i=1}^{n-2} (-1)^{i+1} \dim V_i = \dim(N(T_1)) + (-1)^{n-1} \dim(T_{n-2}(V_{n-2}))$$

By the "rank + nullity" thm. $\dim V_{n-1} = \dim(N(T_{n-1})) + \dim(T_{n-1}(V_{n-1}))$

Add $(-1)^n \dim V_{n-1}$ to both sides of $(*)$, then

$$(**) \quad \sum_{i=1}^{n-1} (-1)^{i+1} \dim V_i = \dim(N(T_1)) + \cancel{(-1)^{n-1} \dim(T_{n-2}(V_{n-2}))} + (-1)^n \dim(T_{n-1}(V_{n-1})) + \cancel{(-1)^n \dim(N(T_{n-1}))}$$

But $\dim T_{n-2}(V_{n-2}) = \dim N(T_{n-1})$ since the sequence is exact, and these enter the formula $(**)$ with different signs. Therefore the statement holds for n. $\square$