Math 1512, **Final Exam Review (Part 1)**

Tue, Dec 14th, 2:50pm-5:40pm, Skiles 271

Topics/keywords:

- Vector algebra: the dot and cross products, norm, angle between two vectors, scalar triple product.
- Linear span, linear independence, basis, dimension.
- Applications: lines/planes in $\mathbb{R}^2$ and $\mathbb{R}^3$, conic sections.
- Linear spaces: $\mathbb{R}^n$ and $\mathbb{C}^n$, functional spaces (e.g., space of polynomials).
- Euclidean spaces: inner product, Cauchy-Schwartz inequality, Gram-Schmidt process.
- Linear maps: kernel(nullspace) and image(range), rank and nullity, one-to-one and onto.
- Matrices: matrix representation of a map with respect to the given bases, space of matrices vs. space of linear maps, Gauss-Jordan method, solving a system of linear equations, inverse matrix.
- Determinants: axioms, computation via Gauss-Jordan, properties (e.g., multilinearity), computation via cofactors.

Be prepared to state theorems, outline algorithms, and define any of the concepts above.
The following are practice questions:

1. A line is given by
$$\begin{cases} x + 2y + 5z &= 1 \\ 3x - 4y - 5z &= 3 \end{cases}$$
   (a) Describe the line in the parametric form $L(P; A) = \{P + tA : t \in \mathbb{R}\}$.
   (b) What is the point of intersection of the given line and the plane $-x + y + z = 0$?
   (c) Describe the set of planes forming the angle of $\pi/3$ with the given line.
2. For which values of a the conic section $x^2 + 2axy + 4y^2 + x + y + 1 = 0$ is
   (a) ellipse?
   (b) hyperbola?
   (c) parabola?
3. Let V be the linear space of functions on defined on $\mathbb{R}$. A linear subspace W is spanned by $\{1, \cos x, \sin x, \cos^2 x, \sin^2 x\}$.
   (a) Find a basis of W.
   (b) Find the coordinates of the function $\cos 2x$ in that basis.
   (c) What is the dimension of W?

4. Prove that for functions f and g continuous on the interval $[a, b]$ the following inequality holds:

$$\left(\int_a^b t^2 f(t)g(t)dt\right)^2 \leq \left(\int_a^b t^2 f(t)^2 dt\right)\left(\int_a^b t^2 g(t)^2 dt\right)$$

5. page 110, #16.

6. Let $V = P_3$, the linear space of polynomial of degree at most 3. Define the map $T : V \to V$ as follows:

$$T(p(x)) = p'(x) + p(x)$$

   (a) Find the matrix A representing the map with respect to the basis $\{1, x, x^2, x^3\}$.

   (b) Is the map T onto? one-to-one?

   (c) Is T invertible? If so, find the inverse of T.

7. Prove that for a square matrix A,

   (a) if $A^2 = 0$ then $I - A$ is invertible.

   (b) if $A^m = 0$ then $I - A$ is invertible, for any positive integer m.

8. Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -2 & 2 \\ -4 & 0 & 2 \end{bmatrix}$$

9. For a parameter $a \in \mathbb{R}$ solve the system

$$\begin{aligned} x - y + z + w &= 4 \\ 2x - y + w &= 6 \\ x + y - z + w &= a \end{aligned}$$

10. page 185, #5.