Chapter 5 of Calculus^{++}: Description and prediction of motion

by

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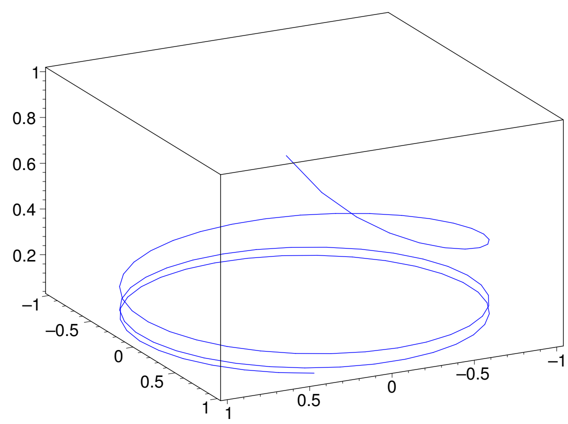
Section 1: Curves in $I\!R^n$

1.1 Functions from $I\!R$ to $I\!R^n$

Consider a function $\mathbf{x}(t)$ of the real variable t with values in $I\!R^n$. For example, let us consider $n = 3$, and

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1/t \end{bmatrix} . \tag{1.1}$$

Here is a three dimensional plot of the curve traced out by $\mathbf{x}(t)$ as t varies from $t = 1$ to $t = 20$:



Such vector valued functions arise whenever we describe the position of a point particle as a function of time. But more generally, we might have any sort of system that is described by n parameters. These could be, for example, the voltages across n points in an electric circuit. We can arrange this data into a vector, and if the data is varying with time, as is often the case in applications, we then have a time dependent vector $\mathbf{x}(t)$ in $I\!R^n$.

When quantities are varying in time, it is often useful to consider their rates of change; i.e., derivatives.

Definition (Derivatives of Vector Valued Functions) Let $\mathbf{x}(t)$ be a vector valued function of the variable t. We say that $\mathbf{x}(t)$ is *differentiable at* $t = t_0$ with derivative $\mathbf{x}'(t_0)$ in case

$$\lim_{h \to 0} \frac{1}{h} \left(\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)\right) = \mathbf{x}'(t_0)$$

in the sense that this limit exists for each of the n entries separately. A vector valued function is *differentiable* in some interval (a, b) if it is differentiable for each t_0 in (a, b).

There is nothing really new going on here. To compute the derivative of $\mathbf{x}(t)$, you just differentiate it entry by entry in the usual way.

Indeed, consider a t dependent vector $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ in $I\!R^2$. Then, by the rules for vector subtraction and scalar multiplication,

$$\begin{aligned} \frac{1}{h}\left(\mathbf{x}(t+h) - \mathbf{x}(t)\right) &= \frac{1}{h}\left(\begin{bmatrix} x(t+h) \\ y(t+h) \end{bmatrix} - \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) \\ &= \begin{bmatrix} (x(t+h) - x(t))/h \\ (y(t+h) - y(t))/h \end{bmatrix} \end{aligned}$$

Now taking the limits on the right, entry by entry, we see that $\mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ provided $x(t)$ and $y(t)$ are both differentiable. The same reduction to single variable differentiation clearly extends to any number of entries.

Example 1 (Computing the derivative of a vector valued function of t) Let $\mathbf{x}(t)$ be given by (1.1). Then for any $t \neq 0$,

$$\mathbf{x}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ -1/t^2 \end{bmatrix} .$$

Because we just differentiate vectors entry by entry without mixing the entries up in any way, familiar rules for differentiating numerically valued functions hold for vector valued functions as well. In particular, the derivative of a sum is still the sum of the derivatives, etc.:

$$(\mathbf{x}(t) + \mathbf{y}(t))' = \mathbf{x}'(t) + \mathbf{y}'(t) \qquad (1.2).$$

Things are only slightly more complicated with the product rule because now we have several types of products to consider. Here is an example that we shall need soon: a "product rule" for the dot product.

Theorem 1 (Differentiating Dot Products) *Suppose that $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are differentiable vector valued functions for t in (a, b) with values in $I\!R^n$. Then $\mathbf{v}(t) \cdot \mathbf{w}(t)$ is differentiable for t in (a, b), and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) \cdot \mathbf{w}(t) = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t) . \qquad (1.3)$$

Proof: For each i, we have by the usual product rule

$$\frac{\mathrm{d}}{\mathrm{d}t} v_i(t) w_i(t) = v_i'(t) w_i(t) + v_i(t) w_i'(t) .$$

Summing on i now gives us (1.3). ■

There are a number of other product rules. For example if $f(t)$ is a differentiable function from $I\!R$ to $I\!R$, and $\mathbf{x}(t)$ is a differentiable curve in $I\!R^n$, then $\mathbf{z}(t) = f(t)\mathbf{x}(t)$ is another differentiable curve in $I\!R^n$, and

$$(f(t)\mathbf{x}(t))' = f'(t)\mathbf{x}(t) + f(t)\mathbf{x}'(t) .$$

This may be established by applying the usual product rule in each coordinate. The details are left as an exercise.

1.2 Differentiability and the tangent line approximation

In the previous subsection, we gave a very simple and direct definition of differentiability for a vector valued function $\mathbf{x}(t)$: It was simple and direct because it made full use of our understanding of derivatives of functions from $I\!R$ to $I\!R$.

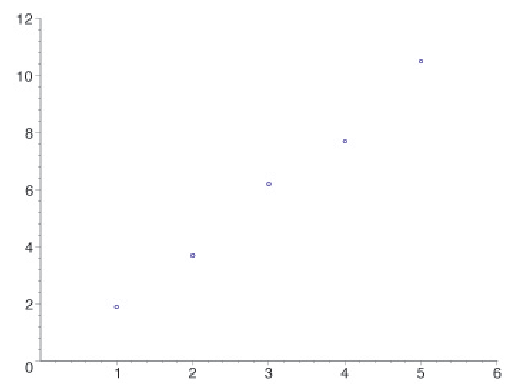
There is another way to look at derivatives of vector values functions $\mathbf{x}(t)$ that is more geometric, and which often provides useful insight: A curve $\mathbf{x}(t)$ is differentiable at $t = t_0$ if and only if looks like *linear motion*

$$\mathbf{x}(t_0) + (t - t_0)\mathbf{v}$$

for some vector $\mathbf{v}$, when t is very close to t_0.

That is, when you "zoom in" on a graph of the curve, so that all you see in your graph is a small segment on which $|t - t_0|$ is small, this segment looks like a straight line segment, traversed at constant speed. This line is called the *tangent line* to the curve at $\mathbf{x}(t_0)$.

Here is a graph of a differentiable curve and its tangent line at a particular point:



You can see from the picture that if you "zoomed in" further, the tangent line and the curve itself could not be distinguished in the graph.

That is, whenever $|t - t_0|$ is *small enough*, but still positive, the distance between the curve $\mathbf{x}(t)$ and the line $\mathbf{x}(t_0) + (t - t_0)\mathbf{v}$ is *so much smaller* that it does not amount to even one pixel in our "zoomed in" graph.

The width and height of the graph are proportional to $|t - t_0|$, since for $|t - t_0| \le r$ the length of the line segment $\mathbf{x}(t_0)+(t-t_0)\mathbf{v}$ is $2|\mathbf{v}|r$. If for $|t-t_0| < r$, $|\mathbf{x}(t)-[\mathbf{x}(t_0)+(t-t_0)\mathbf{v}]|$ is a sufficiently small *percentage** of $2|\mathbf{v}|r$, and hence of the screen width and height, it will not show up as even one pixel.

• *That is, the curve and the line are indistinguishable when*

$$\frac{|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t - t_0)\mathbf{v}]|}{|t - t_0|} \approx 0 \qquad \text{for} \qquad |t - t_0| \approx 0\ .$$

* This is the key: It has not only to be small, but a small *percentage* of the already small width and height of the graph.

We now show that the curve is indistinguishable from a line – the tangent line – in this sense if and only if it is differentiable.

Theorem 2 (Differentiability and the tangent line) *Let* $\mathbf{x}(t)$ *be a function with values in* $I\!R^n$ *that is defined on some open interval* (a,b) *with* $a < t_0 < b$*. Then* $\mathbf{x}(t)$ *is differentiable at* $t = t_0$ *is and only if there is a vector* $\mathbf{v}$ *in* $I\!R^n$ *such that*

$$\lim_{t\to 0} \frac{|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{v}]|}{|t-t_0|} = 0\ . \tag{1.4}$$

In this case, $\mathbf{v} = \mathbf{x}'(t_0)$*.*

Proof: Suppose that (1.4) is true. Let $x_j(t)$ denote the jth component of $\mathbf{x}(t)$ and let v_j denote the jth component of $\mathbf{v}$. Then

$$(x_j(t) - [x_j(t_0) + (t-t_0)v_j]) = (\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{v}]) \cdot \mathbf{e}_j$$

so that by the Schwarz inequality, and the fact that $|\mathbf{e}_j| = 1$,

$$|x_j(t) - [x_j(t_0) + (t-t_0)v_j]| \le |\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{v}]|\ .$$

It now follows from (1.4) that

$$\lim_{t\to 0} \frac{|x_j(t) - [x_j(t_0) + (t-t_0)v_j]|}{|t-t_0|} = 0\ . \tag{1.5}$$

Now let h denote $t - t_0$. Then

$$\left|\frac{x_j(t_0+h) - x_j(t_0)}{h} - v_j\right| = \frac{|x_j(t) - [x_j(t_0) + (t-t_0)v_j]|}{|t-t_0|}\ .$$

It follows that (1.5) is true if and only if for each j,

$$\lim_{h\to 0} \frac{x_j(t_0+h) - x_j(t_0)}{h} = v_j\ .$$

In this case, for each j, $x_j(t)$ is differentiable, and

$$v_j = x_j'(t_0)\ .$$

This shows that if (1.4) is true, then $\mathbf{x}(t)$ is differentiable, and $\mathbf{v} = \mathbf{x}'(t_0)$.

Conversely, suppose that $\mathbf{x}(t)$ is differentiable, and $\mathbf{v} = \mathbf{x}'(t_0)$. Then by the definition of differentiability, (1.5) is true for each j. We have to show that this means that (1.4) is true.

For this purpose, we use the following inequality: For any vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $I\!R^n$,

$$|\mathbf{y}| \le \sum_{j=1}^{n} |y_j| \ . \tag{1.6}$$

Let us accept the validity of (1.6) for the moment, and proceed with the question at hand. (we will prove this in Lemma 1 below.)

By (1.6),

$$|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{v}]| \le \sum_{j=1}^{n} |x_j(t) - [x_j(t_0) + (t-t_0)v_j]| \ .$$

Then of course

$$\frac{|\mathbf{x}(t) - [\mathbf{x}(t_0) + (t-t_0)\mathbf{v}]|}{|t-t_0|} \le \sum_{j=1}^{n} \frac{|x_j(t) - [x_j(t_0) + (t-t_0)v_j]|}{|t-t_0|} \ .$$

Now (1.4) follows from (1.5). ■

We now prove the lemma that grants us the use of (1.6). We single this out s a separate lemma as it will be useful many times in the future. Roughly speaking, what (1.6) tells us is pretty clear: If each entry of a vector $\mathbf{y}$ is small, then $|\mathbf{y}|$ is small. The inequality (1.6) is just a precise quantitative expression of this fact.

Lemma 1 *For any vector* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ *in* $I\!R^n$, $|\mathbf{y}| \le \sum_{j=1}^{n} |y_j|$.

Proof Define C by

$$C = \max\{|y_j| \ : \ j = 1, \dots, n\ \} \ .$$

Then

$$|\mathbf{y}|^2 = \sum_{j=1}^{n} |y_j|^2 \le \sum_{j=1}^{n} C|y_j| = C\left(\sum_{j=1}^{n} |y_j|\right) \ .$$

But the maximum of n non negative quantities is no greater than their sum, so

$$C \le \sum_{j=1}^{n} |y_j| \ .$$

Combining the last two inequalities gives us $|\mathbf{y}|^2 \le \left(\sum_{j=1}^n |y_j|\right)^2$ and hence (1.6). ■

Theorem 2 motivates the following definition:

Definition Suppose that $\mathbf{x}(t)$ is a function with values in $I\!R^n$ that is defined on some open interval (a, b) with $a < t_0 < b$, and $\mathbf{x}(t)$ is differentiable at $t = t_0$.

Then, with $\mathbf{v} = \mathbf{x}'(t_0)$, the approximation

$$\mathbf{x}(t) \approx \mathbf{x}(t_0) + (t - t_0)\mathbf{v}(t_0)$$

is called the *tangent line approximation*, and the parameterized line on the right hand side is called the *tangent line to* $\mathbf{x}(t)$ *at* t_0. The vector $\mathbf{v}(t_0)$ is called the *velocity vector at* $t = t_0$.

Example 2 (The tangent line approximation) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} t \\ 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$. Then for all $t > 0$,

$$\mathbf{x}'(t) = \begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}''(t) = \begin{bmatrix} 0 \\ (2t)^{-1/2} \\ 1 \end{bmatrix} .$$

Taking $t_0 = 1$,

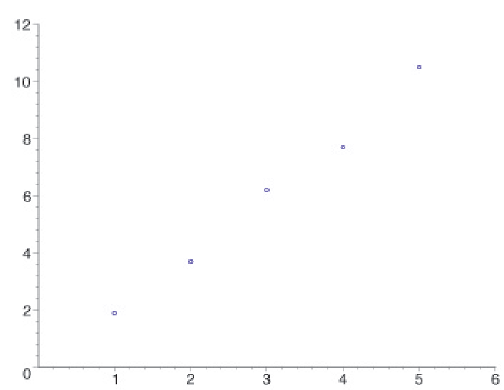
$$\mathbf{x}(1) = \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} \qquad \mathbf{x}'(1) = \begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}''(1) = \begin{bmatrix} 0 \\ 2^{-1/2} \\ 1 \end{bmatrix} .$$

Therefore, when $t \approx 1$, $\mathbf{x}(t) \approx \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + (t-1)\begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} + \dfrac{(t-1)^2}{2}\begin{bmatrix} 0 \\ 2^{-1/2} \\ 1 \end{bmatrix}$.

We get the tangent line by just keeping the linear term in this approximation. Hence the tangent line at $t = 1$ is given in parametric form by

$$\mathbf{z}(t) = \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + (t-1)\begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} .$$

Here is a plot showing the curve for $0 \le t \le 2$, together with the tangent line at $t = 1$.



Observe the tangency. (In this example we have computed the tangent line that we have graphed earlier.) As you can see in the graph, the tangent line fits quite well at $t = t_0$, as it should.

The velocity is a vector quantity, with both magnitude and direction.

Definition The magnitude of the velocity vector is called the *speed*. We denote it by $v(t)$. That is,

$$v(t) = |\mathbf{v}(t)|\ .$$

Provided that $v(t) \neq 0$, we can define a unit vector valued function $\mathbf{T}(t)$ by

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t)\ . \tag{1.7}$$

Then clearly

$$\mathbf{v}(t) = v(t)\mathbf{T}(t)\ . \tag{1.8}$$

The vector $\mathbf{T}(t)$ is called the *unit tangent vector* at time t.

This factorization of the velocity vector into a unit vector giving the direction of motion, and a scalar multiple giving the speed of motion in that direction provides a very good way to think about the motion of point particles, as we shall explain.

Example 3 (Speed and the unit tangent vector) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} t \\ (2t)^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as in the previous example. Then, as we have seen, for all $t > 0$, $\mathbf{x}'(t) = \begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix}$. We then easily compute that

$$v(t) = \sqrt{1 + 2t + t^2} = 1 + t\ ,$$

and so

$$\mathbf{T}(t) = \frac{1}{1+t}\begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix}\ .$$

1.3 Acceleration

Given a differentiable curve $\mathbf{x}(t)$ in $I\!R^n$, we can differentiate, and get another curve in $I\!R^n$, namely $\mathbf{v}(t)$ where $\mathbf{v}(t) = \mathbf{x}'(t)$.

We can now try to differentiate $\mathbf{v}(t)$. This involves a second derivative, which is just what you get when you differentiate twice. That is,

$$\mathbf{x}''(t) = \mathbf{v}'(t)\ .$$

The derivative of the velocity is called the *acceleration*. The vector valued function $\mathbf{x}(t)$ is *twice differentiable* in case each of its entries is twice differentiable in the single variable sense. We usually denote the acceleration by $\mathbf{a}(t)$. That is,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t)\ .$$

Like the velocity, the acceleration is also a vector quantity, with both magnitude and direction. However, we shall find it useful to decompose the acceleration into components that are parallel and perpendicular to the velocity.

Here is one reason: Since $v^2(t) = \mathbf{v}(t)\cdot\mathbf{v}(t)$, we have from Theorem 1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}v^2(t) = \mathbf{a}(t)\cdot\mathbf{v}(t) + \mathbf{v}(t)\cdot\mathbf{a}(t) = 2\mathbf{a}(t)\cdot\mathbf{v}(t)\ .$$

On the other hand, by the product rule, $\dfrac{\mathrm{d}}{\mathrm{d}t}v^2(t) = 2v(t)v'(t)$, and hence

$$2v(t)v'(t) = 2\mathbf{a}(t)\cdot\mathbf{v}(t)\ . \tag{1.9}$$

Hence, if $v'(t) = 0$ if and only if $\mathbf{a}(t)$ is orthogonal to $\mathbf{v}(t)$. Assuming that $v(t) \neq 0$, we can cancel $2v(t)$ from both sides to obtain

$$v'(t) = \mathbf{a}(t)\cdot\mathbf{T}(t)\ . \tag{1.10}$$

Notice that the speed can be constant without the velocity being constant. This is what happens in steady circular motion. The direction of the velocity keeps changing though the speed is constant.

Example 4 (Steady circular motion) Let $\mathbf{x}(t)$ be given by

$$\mathbf{x}(t) = r\begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix}$$

for some $r > 0$. Then,

$$\mathbf{v}(t) = r\begin{bmatrix}-\sin(t)\\ \cos(t)\end{bmatrix} \qquad \text{and} \qquad \mathbf{a}(t) = r\begin{bmatrix}-\cos(t)\\ -\sin(t)\end{bmatrix}\ .$$

Then $v(t) = |\mathbf{v}(t)| = r$, and $\mathbf{T}(t) = (1/v(t))\mathbf{v}(t) = \begin{bmatrix}-\sin(t)\\ \cos(t)\end{bmatrix}$.

Also, as you see, the velocity is constantly changing although the speed is constant: $v(t) = r$ for all t. Also, as you can compute,

$$\mathbf{a}(t)\cdot\mathbf{v}(t) = 0$$

for all t.

We can gain more insight into the nature of acceleration by considering circular motion at a variable rate:

Example 5 (Circular motion at a variable rate) Let $\mathbf{x}(t)$ be given by

$$\mathbf{x}(t) = r\begin{bmatrix}\cos(\theta(t))\\ \sin(\theta(t))\end{bmatrix}$$

for some $r > 0$, and some function $\theta(t)$ giving the angular coordinate of the point at time t. Let us suppose that $\theta(t)$ is twice differentiable. Then,

$$\mathbf{v}(t) = r\theta'(t)\begin{bmatrix}-\sin(\theta(t))\\ \cos(\theta(t))\end{bmatrix} \qquad \text{and} \qquad \mathbf{a}(t) = r\theta''(t)\begin{bmatrix}-\sin(\theta(t))\\ \cos(\theta(t))\end{bmatrix} + r(\theta'(t))^2\begin{bmatrix}-\cos(\theta(t))\\ -\sin(\theta(t))\end{bmatrix}\ .$$

Then

$$v(t) = |\mathbf{v}(t)| = r\theta'(t) \qquad \text{and} \qquad \mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \begin{bmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix} .$$

As you see, $v'(t) = (r\theta'(t)) = r\theta''(t)$, and that $r(\theta'(t))^2 = v^2(t)/r$. Therefore, we can rewrite the acceleration vector as

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \frac{v^2(t)}{r}\mathbf{N}(t) \tag{1.11}$$

where $\mathbf{N}(t)$ denotes the unit vector

$$\mathbf{N}(t) = \begin{bmatrix} -\cos(\theta(t)) \\ -\sin(\theta(t)) \end{bmatrix} .$$

Notice that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$ for all t, so that (1.11) gives a decomposition of the acceleration into two orthogonal components: $v'(t)\mathbf{T}(t)$ is the component parallel to the direction of motion, and $\frac{v^2(t)}{r}\mathbf{N}(t)$ is the component perpendicular to the direction of motion.

Notice that $v'(t)$, the rate of change of the speed, only enters into the first component in (1.11). If you are driving a car around a circular track, and step on the gas, or on the brakes, you will feel an acceleration in the direction of motion, pushing you back in your seat, or up against the seatbelt. That is the tangential component of the acceleration.

On the other hand, even if you keep your speed constant, you will still feel an acceleration due to your constant turning. You will experience this acceleration as a so–called "centrifugal force", pushing you away from the center of the track. This higher your speed, and the smaller the radius of the track, the greater this acceleration will be. Notice that this agrees with the formula above, according to which the magnitude of the orthogonal component of the acceleration is $v^2(t)/r$.

This decomposition is useful even when the track is not circular. This leads us to the following definitions:

Definition (Normal and tangential acceleration, normal vector, and curvature) Let $\mathbf{x}(t)$ be a twice differentiable parameterized curve in $I\!R^n$. Let

$$\mathbf{a}(t) = \mathbf{a}_{\|}(t) + \mathbf{a}_{\perp}(t)$$

be the decomposition of the acceleration into its components parallel and orthogonal to $\mathbf{v}(t)$, assuming that $\mathbf{v}(t) \neq 0$, so that this decomposition is well defined. Then $\mathbf{a}_{\|}(t)$ is called the *tangential acceleration*, and $\mathbf{a}_{\perp}(t)$ is called the *normal acceleration*.

Furthermore, in case $\mathbf{a}_{\perp}(t) \neq 0$, we define the *unit normal vector* $\mathbf{N}(t)$ by

$$\mathbf{N}(t) = \frac{1}{|\mathbf{a}_{\perp}(t)|}\mathbf{a}_{\perp}(t) \ . \tag{1.12}$$

Finally, we define the *curvature* at time t, $\kappa(t)$, by

$$\kappa(t) = \frac{|\mathbf{a}_{\perp}(t)|}{v^2(t)} \tag{1.13}$$,

and the *radius of curvature* at time t by

$$\rho(t) = \frac{1}{\kappa(t)} \tag{1.14}$$,

It is important to notice that $\mathbf{N}(t)$ points in the same direction as $\mathbf{T}'(t)$, the derivative of $\mathbf{T}(t)$. Thus, it points in the direction in which the curve is turning. To see this, let us notice first of all that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$: Since $\mathbf{T}(t)$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1 \ .$$

Therefore, by Theorem 1,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}1 = 2\mathbf{T}(t) \cdot \mathbf{T}'(t) \ .$$

To go on and show that $\mathbf{T}'(t)$ actually points in the same direction as $\mathbf{N}(t)$, we compute the acceleration using $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Then

$$\mathbf{a}(t) = (v(t)\mathbf{T}(t))' = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \ . \tag{1.15}$$

Notice that the first term on the right is a multiple of $\mathbf{T}(t)$, and the second term on the right is orthogonal to $\mathbf{T}(t)$ by what we have just seen. Therefore, (1.15) is the decomposition of the acceleration into its tangential and orthogonal components. That is,

$$(\mathbf{a}(t))_{\|} = v'(t)\mathbf{T}(t) \qquad \text{and} \qquad (\mathbf{a}(t))_{\perp} = v(t)\mathbf{T}'(t) \ . \tag{1.16}$$

But by (1.12) and (1.13),

$$(\mathbf{a}(t))_\perp = v^2(t)\kappa(t)\mathbf{N}(t) \ . \tag{1.17}$$

Comparing (1.15), (1.16) and (1.17), we deduce two useful formulas, that deserve to be enshrined in a theorem:

Theorem 3 *Let* $\mathbf{x}(t)$ *be a twice diffeentiable curve in* $I\!R^n$*. Then*

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v^2(t)\kappa(t)\mathbf{N}(t) \ , \tag{1.18}$$

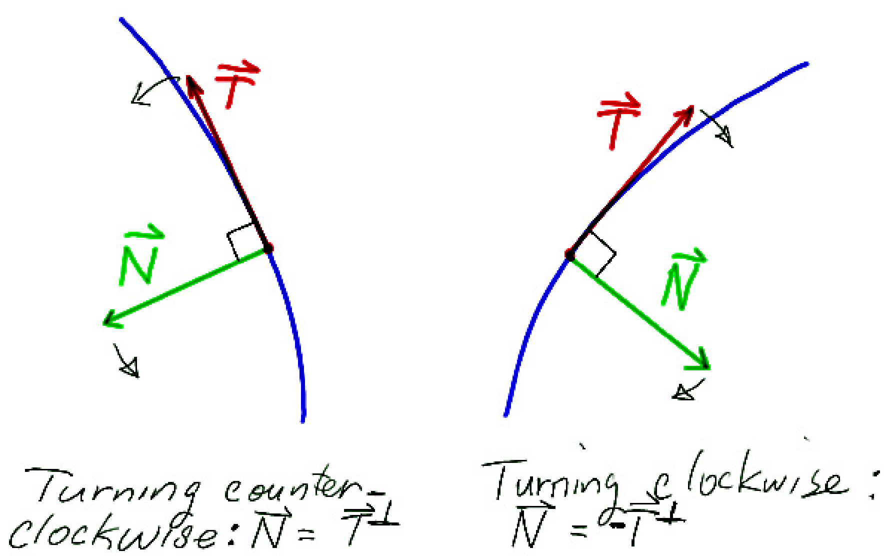
and

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t) \ . \tag{1.19}$$

Proof: simply compare (1.15), (1.16) and (1.17). ■

The formula (1.18) should be compared with (1.11). This comparison explains the justification for referring to $\rho(t) = 1/\kappa(t)$ as the "radius of curvature" of time t.

In $I\!R^2$, there are just two directions perpendicular to $\mathbf{T}$, namely $\mathbf{T}^\perp$ and $-\mathbf{T}^\perp$. Since $\mathbf{N}$ is perpendicular to $\mathbf{T}$, either $\mathbf{N} = \mathbf{T}^\perp$ or else $\mathbf{N} = -\mathbf{T}^\perp$. Which it is depends on whether the turning is clockwise or counterclockwise. The formula (1.19) shows that $\mathbf{N}(t)$ points in the same direction as $\mathbf{T}'(t)$: i.e., $\mathbf{N}$ points in the "direction of turning". If, as you drive along a track, you are turning left, then $\mathbf{N} = \mathbf{T}^\perp$, and if you are turning right, then $\mathbf{N} = -\mathbf{T}^\perp$.



Example 6 (Computing the normal and tangential acceleration) Let $\mathbf{x}(t) = \begin{bmatrix} t \\ (2t)^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as before. Then for all $t > 0$, as we have computed before $v(t) = 1 + t$ so that $v'(t) = 1$. We have computed in Example 3 that

$$v(t) = 1 + t \qquad \text{and} \qquad \mathbf{T}(t) = \frac{1}{1+t}\begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix} \ .$$

Therefore, $v'(t) = 1$, and so $\mathbf{a}_{\|}(t) = \mathbf{T}(t)$. Thus,

$$\mathbf{a}_{\|}(t) = \frac{1}{1+t}\begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix} .$$

This is the tangential component of the acceleration. The normal component must be $\mathbf{a}(t)$ minus this. Since we have computed in Example 1 that

$$\mathbf{a}(t) = \mathbf{x}''(t) = \begin{bmatrix} 0 \\ (2t)^{-1/2} \\ 1 \end{bmatrix} ,$$

the normal component is

$$\begin{bmatrix} 0 \\ (2t)^{-1/2} \\ 1 \end{bmatrix} - \frac{1}{1+t}\begin{bmatrix} 1 \\ (2t)^{1/2} \\ t \end{bmatrix} = \frac{1}{1+t}\begin{bmatrix} -1 \\ (1-t)(2t)^{-1/2} \\ 1 \end{bmatrix} .$$

From here we compute

$$|\mathbf{a}_{\perp}(t)| = \frac{1}{\sqrt{2t}} .$$

Hence

$$\mathbf{N}(t) = \frac{\sqrt{2t}}{1+t}\begin{bmatrix} -1 \\ (1-t)(2t)^{-1/2} \\ 1 \end{bmatrix}$$

and

$$\kappa(t) = \frac{\sqrt{2t}}{(1+t)^2} \qquad \text{and} \qquad \rho(t) = \frac{(1+t)^2}{\sqrt{2t}} .$$

1.4 The tangent circle, and the plane of motion

Let $\mathbf{u}_1$ and and $\mathbf{u}_2$ be any given pair of orthogonal unit vectors in $I\!R^n$, and let ρ be any positive number. Consider the parameterized curve $\mathbf{z}(t)$ given by

$$\mathbf{z}(t) = \rho(\cos(t)\mathbf{u}_1 + \sin(t)\mathbf{u}_2) .$$

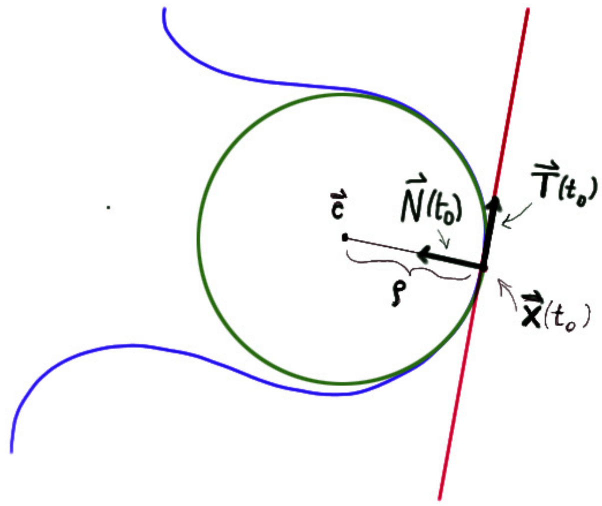
Then, since $\mathbf{u}_1$ and and $\mathbf{u}_2$ are orthogonal unit vectors, $|\mathbf{z}(t)| = \rho$ for all t. Also clearly, $\mathbf{z}(t)$ lies in the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ for all t. The curve $\mathbf{z}(t)$ describes steady circular motion on the circle of radius ρ about the origin in this plane.

To get a more general sort of circular motion, we can shift the center of the circle away from the origin, and replace t by $\theta(t)$ so that the motion on the circle proceeds at a variable rate. If $\mathbf{c}$ is the new center of the circle, we then have

$$\mathbf{z}(t) = \mathbf{c} + \rho(\cos(\theta(t))\mathbf{u}_1 + \sin(\theta(t))\mathbf{u}_2) . \tag{1.20}$$

We are now going if $\mathbf{x}(t)$ is very a twice differentiable curve at $t = t_0$, and if the curvature $\kappa(t_0) \neq 0$, the for t close to t_0, the curve $\mathbf{x}(t)$ is well approximated by such a circular motion, and the circle that "fits best" is uniquely determined. This will be the *tangent circle*. Of course, at a point where the

Here is a graph showing a curve in the plane, together with its the tangent line, and its tangent circle at a particular point $\mathbf{x}(t_0)$.



As you can see, the tangent circle gives a much better fit to the curve than does the tangent line. The point $\mathbf{c}$ is the center of the circle, and ρ is its radius. As we shall see, $rho = \rho(t_0)$, the radius of curvature of the curve $\mathbf{x}(t)$ at $t = t_0$.

Here is one way to think about what the tangent circle represents: If $\mathbf{x}(t)$ is your position at time t as you drive along some track,* you would "feel" at each instant of time t as if you were diving on a circular track of radius $\rho(t)$ with a center at a point $\mathbf{c}(t)$ in the plane spanned by $\mathbf{v}(t)$ and $\mathbf{a}(t)$, provided these vectors are linearly independent. This "instantaneous circular track" is the tangent circle.

The tangent circle is only well defined when the curvature $\kappa(t_0)$ is not zero, so that $\mathbf{N}(t_0)$ is well defined. It will lie in the plane through $\mathbf{x}(t_0)$ spanned by $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$. This plane is called the *plane of motion*. Here are the formal definitions:

Definition (Tangent circle and plane of motion) Let $\mathbf{x}(t)$ be a twice differentiable curve. Then at each t_0 for which $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$ is a linearly independent set of vectors, the *tangent circle* to this curve is the circle with center

$$\mathbf{c}(t_0) = \mathbf{x}(t_0) + \rho(t_0)\mathbf{N}(t_0) \tag{1.21}$$

with radius $\rho(t_0)$, lying in the plane through $\mathbf{x}(t_0)$ spanned by $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$. This plane is called the *instantaneous plane of motion*.

We will soon prove that the formula given in this definition does indeed give the circle that "best fits" the curve at $\mathbf{x}(t_0)$. First, let us accept this, and use the formula to compute and graph a tangent circle.

* This makes sense for $n = 2$ or 3, but the formulas we will deduce are independent of the interpretation, and are valid for all $n \geq 2$.

Example 7 (Computing the tangent circle) Let $\mathbf{x}(t) = \begin{bmatrix} t \\ (2t)^{3/2}/3 \\ t^2/2 \end{bmatrix}$. In Example 2, we have computed the tangent line at $t_0 = 1$, and found it to be given by

$$\begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + (t-1)\begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} .$$

In Example 6, we computed $\mathbf{N}(t)$, $\mathbf{T}(\mathrm{t})$ and $\rho(t)$. Evaluating them at $t = 1$, we get

$$\mathbf{T}(1) = \frac{1}{2}\begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} \qquad \mathbf{N}(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \rho(1) = 2^{3/2}.$$

Therefore,

$$\mathbf{c}(1) = \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2^{3/2}/3 \\ 5/2 \end{bmatrix} .$$

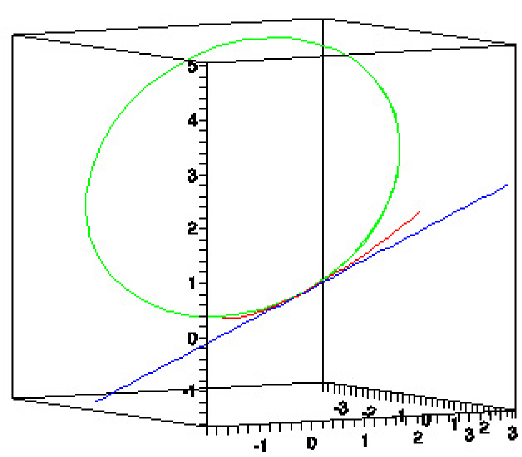
Hence the tangent circle is parameterized by

$$\mathbf{z}(\theta) = \begin{bmatrix} -1 \\ 2^{3/2}/3 \\ 5/2 \end{bmatrix} - \cos(\theta)2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \sin(\theta)\sqrt{2}\begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} ,$$

with

$$\theta = 2(t-t_0) + (t-t_0)^2/2 .$$

Here is a graph showing the curve itself, the tangent line, and the tangent circle at $t_0 = 1$.



As you can see, the tangent circle gives a much better fit than the tangent line at the point of tangency. Our next goal is to justify the formula for the tangent circle that we have given in the definition, and have just used in the example.

To get a formula for the tangent circle, we are going to show that for each t_0, we can choose $\mathbf{c}$, ρ, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\theta(t)$ in (1.20) so that

$$\lim_{t \to t_0} \frac{|\mathbf{x}(t) - \mathbf{z}(t)|}{|t-t_0|^2} = 0 . \qquad (1.22)$$

Moreover, as we shall see, there is just one way to choose $\mathbf{c}$, ρ, $\mathbf{u}_1$ and $\mathbf{u}_2$ to get such a good approximation. Since $\mathbf{c}$, ρ, $\mathbf{u}_1$ and $\mathbf{u}_2$ are what determines the circle, this will mean that it makes sense to talk about *the* tangent circle.

First, let us match the speeds, We readily compute

$$\mathbf{z}'(t) = \rho\theta'(t)(-\sin(\theta(t))\mathbf{u}_1 + \cos(\theta(t))\mathbf{u}_2) \ . \tag{1.23}$$

so that $|\mathbf{z}'(t)| = \rho\theta'(t)$. To match the speeds, we require that $\rho\theta'(t) = v(t)$ where of course $v(t) = |\mathbf{x}'(t)|$. This gives us a formula for $\theta'(t)$, namely

$$\theta'(t) = \frac{v(t)}{\rho} \ . \tag{1.24}$$

Integrating from t_0 to t gives us a formula for $\theta(t)$ itself:

$$\theta(t) = \frac{1}{\rho}\int_{t_0}^{t} v(r)\mathrm{d}r \ . \tag{1.25}$$

Now that the speeds have been matched, let us match the rest, From (1.23) and (1.24), since $\theta(t_0) = 0$,

$$\mathbf{z}'(t_0) = v(t_0)\mathbf{u}_2 \ .$$

Of course, $\mathbf{x}(t_0) = v(t_0)\mathbf{T}(t_0)$, so we get $\mathbf{z}'(t_0) = \mathbf{x}'(t_0)$ is and only if

$$\mathbf{u}_2 = \mathbf{T}(t_0) \ . \tag{1.26}$$

To go further, we math the acceleration vectors: We compute

$$\mathbf{z}'(t) = \rho\theta''(t)(-\sin(\theta(t))\mathbf{u}_1 + \cos(\theta(t))\mathbf{u}_2) - \rho(\theta'(t))^2(\cos(\theta(t))\mathbf{u}_1 + \sin(\theta(t))\mathbf{u}_2) \ .$$

Evaluating this at t_0, and using $\theta(t_0) = 0$, we get

$$\mathbf{z}'(t_0) = \rho\theta''(t)\mathbf{u}_2 - \rho(\theta'(t))^2\mathbf{u}_1 \ .$$

Using (1.24) and (1.26), this is

$$\mathbf{z}'(t_0) = v'(t_0)\mathbf{T}(t_0) - \frac{v^2(t_0)}{\rho}\mathbf{u}_1 \ .$$

Comparing this with

$$\mathbf{x}''(t_0) = v'(t_0)\mathbf{T}(t_0) - \frac{v^2(t_0)}{\rho(t_0)}\mathbf{N}(t_0) \ ,$$

we see that to match the acceleration vectors, we must take $\rho = \rho(t_0)$ and $\mathbf{u}_1 = -\mathbf{N}(t_0)$.

Finally, to determine $\mathbf{c}$, we require that $\mathbf{z}(t_0) = \mathbf{x}(t_0)$. Since

$$\mathbf{z}(t_0) = \mathbf{c} + \rho\mathbf{u}_2 = \mathbf{c} - \rho(t_0)\mathbf{N}(t_0) \ ,$$

we require that

$$\mathbf{c} = \mathbf{x}(t_0) + \rho(t_0)\mathbf{N}(t_0) \ .$$

In summary, with the speeds matched through (1.25) we have

$$\begin{aligned} \mathbf{z}(t_0) &= \mathbf{x}(t_0) \\ \mathbf{z}'(t_0) &= \mathbf{x}'(t_0) \\ \mathbf{z}''(t_0) &= \mathbf{x}''(t_0) \ . \end{aligned} \tag{1.27}$$

if and only if

$$\mathbf{z}(t) = \mathbf{c} + \rho\Big(-\cos(\theta(t))\mathbf{N} + \sin(\theta(t))\mathbf{T}\Big) \tag{1.28}$$

Let us fix this well matched choice for $\mathbf{z}(t)$. Then, for each $j = 1, \ldots, n$, since $z_j(t_0) = x_j(t_0)$, $z_j'(t_0) = x_j'(t_0)$ and $z_j''(t_0) = x_j''(t_0)$, so that both $x_j(t)$ and $z_j(t)$ have the same second order Taylor polynomial at $t = t_0$. Therefore,

$$\lim_{t\to 0} \frac{|z_j(t) - x_j(t)|}{|t - t_0|^2} = 0 \ . \tag{1.29}$$

From this and the lemma following Theorem 2, we see that

$$\lim_{t\to t_0} \frac{|\mathbf{z}(t) - \mathbf{x}(t)|}{|t - t_0|^2} = 0 \ . \tag{1.30}$$

Conversely, if (1.30) is true, then so is (1.29) for each j. Consequently, both $x_j(t)$ and $z_j(t)$ must have the same second order Taylor polynomial at $t = t_0$. Therefore, eqv(vcc94) must be true. We have proved the following theorem:

Theorem 4 (Tangent circle approximation) *Let $\mathbf{x}(t)$ be a parameterized curve in $I\!R^n$ that is twice differentiable at $t = t_0$. Let $\mathbf{z}(t)$ be the parameterized circle given by (1.20), and with $\theta(t)$ given by (1.25). Then (1.30) is true if and only if $\mathbf{u}_2 = \mathbf{T}(t_0)$, $\mathbf{u}_1 = -\mathbf{N}(t_0)$, $\rho = \rho(t_0)$ and $\mathbf{c} = \mathbf{x}(t_0) + \rho(t_0)\mathbf{N}(t_0)$. where the quantities on the right hand sides are computed for $\mathbf{x}(t)$.*

In particular, there is just one circular track that fits so well that (1.30) is true, and the radius of this track is $\rho(t_0)$, which further justifies the use of the term "radius of curvature" for this quantity.

1.5 Torsion and the Frenet–Seret formulae for a curve in $I\!R^3$

The case in which $n = 3$ is especially important, since we live in a three dimensional world. In this case, we can compute a unit normal to the plane of motion by taking the cross product of $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$.

Definition (Binormal vector) Let $\mathbf{x}(t)$ be a twice differentiable curve in $I\!R^3$. Then at each t_0 for which $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$ is a linearly independent set of vectors, so that $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ are well defined. Then the *binormal vector* $\mathbf{B}(t_0)$ is defined by

$$\mathbf{B}(t_0) = \mathbf{T}(t_0) \times \mathbf{N}(t_0) \ . \tag{1.31}$$

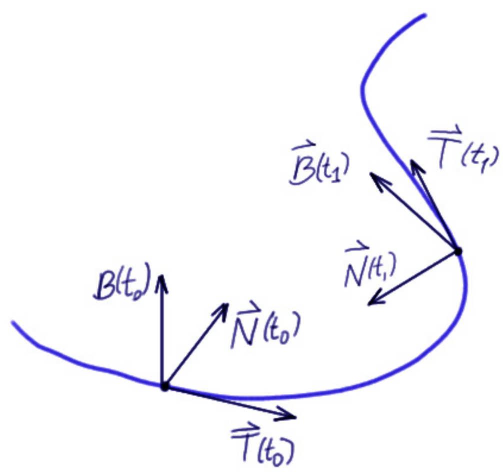
Since $\mathbf{B}(t_0)$ is orthogonal to the plane of motion, and $\mathbf{x}(t_0)$ lies in this plane, the equation for the plane of motion in $I\!R^3$ is given by

$$\mathbf{B}(t_0) \cdot (\mathbf{x} - \mathbf{x}(t_0)) = 0 \ .$$

Let $\mathbf{x}(t)$ be a twice differentiable curve in$I\!R^3$. Then for each time t, the vectors

$$\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$$

are a right handed orthonormal basis of $I\!R^3$. That is, a curve in $I\!R^3$ carries around with itself a special orthonormal basis. How does this basis change with time?



Since $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a basis, we can express each of $\mathbf{T}'(t)$, $\mathbf{N}'(t)$ and $\mathbf{B}'(t)$ as linear combinations of these basis elements. Indeed, we have already seen that

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t) \ . \tag{1.32}$$

Next, consider $\mathbf{B}'(t)$. Since for each t, $\mathbf{B}(t) \cdot \mathbf{T}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}(t) \cdot \mathbf{T}(t)) = \mathbf{B}'(t) \cdot \mathbf{T}(t) + \mathbf{B}(t) \cdot \mathbf{T}'(t) \ .$$

But

$$\mathbf{B}(t) \cdot \mathbf{T}'(t) = \mathbf{B}(t) \cdot (v(t)\kappa(t)\mathbf{N}(t)) = 0 \ ,$$

and so

$$\mathbf{B}'(t) \cdot \mathbf{T}(t) = 0 \ .$$

Also, since for each t, $\mathbf{B}(t) \cdot \mathbf{B}(t) = 1$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}(t) \cdot \mathbf{B}(t)) = 2\mathbf{B}'(t) \cdot \mathbf{B}(t) \ ,$$

so that

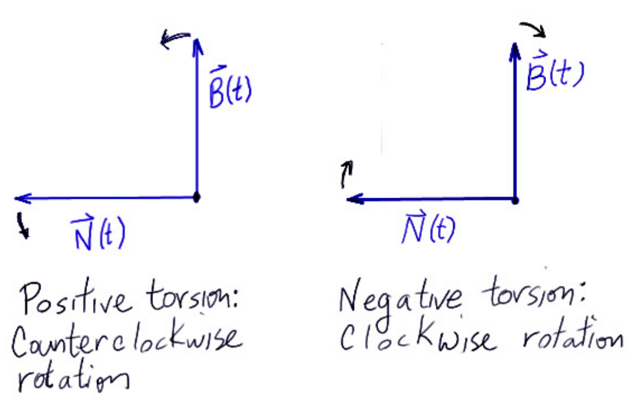
$$\mathbf{B}'(t) \cdot \mathbf{B}(t) = 0 \ .$$

Since $\mathbf{B}'(t)$ has no component in the directions of $\mathbf{T}(t)$ or $\mathbf{B}(t)$, it follows that $\mathbf{B}'(t)$ is a multiple of $\mathbf{N}(t)$. This multiple deserve a name. Therefore, in analogy with (1.32), we define the *torsion* $\tau(t)$ by $\mathbf{B}'(t) = v(t)\tau(t)\mathbf{N}(t)$.

Definition (Torsion) Let $\mathbf{x}(t)$ be a twice differentiable curve in $I\!R^3$. Then at each t_0 for which $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$ is a linearly independent set of vectors, so that $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ are well defined for t in a neigborhood of $t = t_0$. Then the *torsion* at $t = t_0$ is the quantity $\tau(t_0)$ defined by

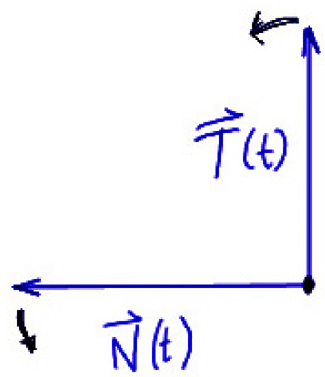
$$\mathbf{B}'(t) = v(t)\tau(t)\mathbf{N}(t) \ . \tag{1.33}$$

The torsion describes the instantaneous rate at which the plane of motion rotates about the tangnet line: If one looks down along the tangen line in the direction of $\mathbf{T}(t_0)$, and there is positive torsion, one sees the binormal vector turning counterclockwise.



Again, in this pcture, we are looking along the tangent line, in the direction of motion.

It is also helpful to consider a "top view", looking down on the plane of motion. Then we see $\mathbf{T}(t)$ and $\mathbf{N}(t)$. The curvature describes the rate at which these vectors rotate. By definition, $\mathbf{N}$ give the direction in which $\mathbf{T}$ is turning, and so the curvature is always non negative, and $\mathbf{T}$ turns towards $\mathbf{N}$.



Finally, let us derive a formula for $\mathbf{N}'(t)$. Looking at the graphs above, you might be able to deduce that

$$\mathbf{N}'(t) = -v(t)\kappa(t)\mathbf{T}(t) - v(t)\tau(t)\mathbf{B}(t)\ . \tag{1.34}$$

Let us deduce this by computational means.

First, since for each t, $\mathbf{N}(t) \cdot \mathbf{N}(t) = 1$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{N}(t)) = 2\mathbf{N}'(t) \cdot \mathbf{N}(t)\ ,$$

so that

$$\mathbf{N}'(t) \cdot \mathbf{N}(t) = 0\ . \tag{1.35}$$

Second, since for each t, $\mathbf{N}(t) \cdot \mathbf{T}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{T}(t)) = \mathbf{N}'(t) \cdot \mathbf{T}(t) + \mathbf{N}(t) \cdot \mathbf{T}'(t)\ .$$

But

$$\mathbf{N}(t) \cdot \mathbf{T}'(t) = \mathbf{N}(t) \cdot (v(t)\kappa(t)\mathbf{N}(t)) = v(t)\kappa(t)\ ,$$

and so

$$\mathbf{N}'(t) \cdot \mathbf{T}(t) = -v(t)\kappa(t)\ , \tag{1.36}$$

Third, since for each t, $\mathbf{N}(t) \cdot \mathbf{B}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{B}(t)) = \mathbf{N}'(t) \cdot \mathbf{B}(t) + \mathbf{N}(t) \cdot \mathbf{B}'(t)\ .$$

But

$$\mathbf{N}(t) \cdot \mathbf{B}'(t) = \mathbf{N}(t) \cdot (v(t)\tau(t)\mathbf{N}(t)) = v(t)\tau(t)\ ,$$

and so

$$\mathbf{N}'(t) \cdot \mathbf{B}(t) = -v(t)\tau(t)\ , \tag{1.37}$$

Combining (1.35), (1.36) and (1.37), we see that (1.34) is true.

Summarizing the results, we have proved the following:

Theorem 5 (Frenet–Seret formulae) *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $I\!R^3$*. Then at each* t_0 *for which* $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$ *is a linearly independent set of vectors, so that* $\mathbf{T}(t)$*,* $\mathbf{N}(t)$ *and* $\mathbf{B}(t)$ *are well defined for* t *in a neigborhood of* $t = t_0$*. Then fro* t *in this neighborhood,*

$$\begin{aligned}
\mathbf{T}'(t) &= v(t)\kappa(t)\mathbf{N}(t) \\
\mathbf{N}'(t) &= -v(t)\kappa(t)\mathbf{T}(t) - v(t)\tau(t)\mathbf{B}(t) \\
\mathbf{B}'(t) &= v(t)\tau(t)\mathbf{N}(t) \ .
\end{aligned}$$

There is a convenient way to combine these three formulae into one. If we define the orthogonal matrix $U(t)$ by

$$U(t) = [\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)]$$

and the antisymmetrix matrix $A(t)$ by

$$A(t) = \begin{bmatrix} 0 & -\kappa(t) & 0 \\ \kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \ ,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = v(t)A(t)U(t) \ .$$

1.6 Reconstructing a curve in $I\!R^3$ given its speed, curvature and torsion

Let $\mathbf{x}(t)$ be a twice differentiable curve in$I\!R^3$. We have already seen how to compute $v(t)$, $\kappa(t)$ and $\tau(t)$.

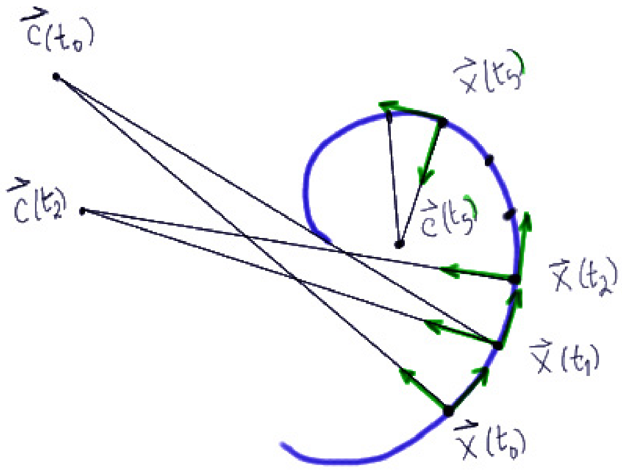
It turns out that this can be reversed: If you know the lists of data $\begin{bmatrix} v(t) \\ \kappa(t) \\ \tau(t) \end{bmatrix}$ for t in some interval $[0, t_*]$ of time, and curvature is never zero for t in $[0, t_*]$, then you can reconstruct the curve itself; i.e., $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, in this time interval, provided you are also given $\mathbf{x}(0)$, $\mathbf{T}(0)$ and $\mathbf{N}(0)$. Once you understand how to do this, you will certainly understand what curvature and torsion are.

Since the x, y plane is a subset of $I\!R^3$, every plane curve in $I\!R^2$ can also be viewed as curve in $I\!R^3$; we just identify $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix}$. For such a curve, the plane of motion is the x, y plane, and $\mathbf{B}(t) = \pm\mathbf{e}_3$. There is therefore no torsion.

It is a little easier to understand this special case, so we start here, with a plane curve. We are given $\mathbf{x}(0)$, $\mathbf{T}(0)$ and $\mathbf{N}(0)$, together with $v(t)$ and $\kappa(t)$ for t in the interval $[0, t_*]$.

The strategy will be a familiar one: We will build up an approximation to the curve by putting together many small pieces. Then we shall take a limit in which the size of the small pieces tends to zero. In this limit, we recover the curve exactly.

Here is a picture of a plane curve that is "spiraling in". The positions $\mathbf{x}(t_j)$ are marked for $j = 0, 1, \ldots, 5$. For t_0, t_1 and t_5, you see indicated that center of the tangent circles $\mathbf{c}(t_0)$, $\mathbf{c}(t_1)$ and $\mathbf{c}(t_5)$ at these times.



Notice that as the curve spirals in, the curvature increases, and so the radii of the tangent circles decreases.

• *As you can see, the curve fits very well to the short circular arcs along these tangent circles. We are going to build up the curve out of such small circular arcs, keeping track of how* $\mathbf{T}$ *and* $\mathbf{N}$ *change as we move along the arcs.*

To do this, pick a number N, and let $h = t_*/N$. For $j = 0, \ldots, N$, define t_j to be jh. On each time interval $t_{j-1} \le t \le t_j$, we will replace use a circular aprroximation to $\mathbf{x}(t)$, and obtain the curve $\mathbf{x}_{(h)}(t)$.

Here is how this works on the first interval $t_0 = 0 \le t \le h = t_1$. From the given data, we know that the initial tangent circle has radius $1/\kappa(0)$, and is centered at

$$\mathbf{c}(0) = \mathbf{x}_{(}0) + (1/\kappa(0))\mathbf{N}(0) \ . \tag{1.38}$$

Our approximate motion will simply be motion along the tangent circle at constant speed $v(0)$. From (1.25), we see that this mean that the angle $\theta(t)$ on the circle will be given by

$$\theta(t) = \kappa(0)v(0)t \ .$$

Using this and (1.38) in (1.28), we get a formula for $\mathbf{x}_{(h)}(t)$ valid for $0 \le t \le h$:

$$\mathbf{x}_{(h)}(t) = [\mathbf{x}(0) + (1/\kappa(0))\mathbf{N}(0)] + \frac{1}{\kappa(0)}[-\cos(\kappa(0)v(0)t)\mathbf{N}(0) + \sin(\kappa(0)v(0)t)\mathbf{T}(0)] \ .$$

Now, using this formula, we can compute $\mathbf{x}_{(h)}(t_1)$, and its unit tangnet and normal vectors at time t_1: $\mathbf{T}_{(h)}(t_1)$ and $\mathbf{N}_{(h)}(t_1)$. We can then repeat the procedure. Just start from $\mathbf{T}_{(h)}(t_1)$ and $\mathbf{N}_{(h)}(t_1)$ as the initial data, and make the same tangent circle approximation for $t \leq t \leq t_2$. If one continues for N steps, one gets the whole curve $\mathbf{x}_{(h)}$ which will be continuous, and piecewise twice differentiable. (The curve will be continuous at each t_k; the circles fit together there, but it will not be continuously differentiable at such times, since in general the speed will change discontinuously there.)

Let us formalize this as a recursive algorithm for generating $\mathbf{x}_{(h)}(t)$ for $0 \leq t \leq t_*$:

Curve contruction algorithm: We are given $\mathbf{x}(0)$, $\mathbf{T}(0)$ and $\mathbf{N}(0)$, as well as $v(t)$ and $\kappa(t)$ for $0 \leq t \leq t_*$. We pick a (large) value of N, and define $h = t_*/N$. We assume that $\kappa(t) > 0$ for all $0 \leq t \leq t_*$.

(1) (initialization) Define $\mathbf{x}_{(h)}(0) = \mathbf{x}(0)$, and define $\mathbf{T}_1 = \mathbf{T}(0)$ and $\mathbf{N}_0 = \mathbf{N}(0)$.

Then, starting with $j = 0$, and **while** $j < N$:
(2) (segment contruction) Given $\mathbf{x}_{(h)}(t_j)$ and the orthonormal unit vectors $\mathbf{T}_j$ and $\mathbf{N}_j$, define the center $\mathbf{c}_j$ by

$$\mathbf{c}_j = \mathbf{x}_{(h)}(t_j) + (1/\kappa(t_j))\mathbf{N}_j \ , \tag{1.39}$$

and define $\mathbf{x}_{(h)}$ for $t_j \leq t \leq t_{j+1}$ by

$$\mathbf{x}_{(h)}(t) = \mathbf{c}_j + \frac{1}{\kappa(t_j)}\big(-\cos[\kappa(t_j)v(t_j)(t-t_j)]\mathbf{N}_j + \sin[\kappa(t_j)v(t_j)(t-t_j)]\mathbf{T}_j\big) \ . \tag{1.40}$$

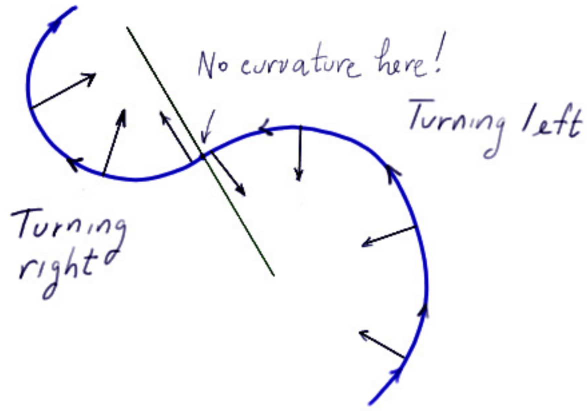
(3) (updating $\mathbf{T}$ *and* $\mathbf{N}$*)* Using (1.40) compute $\mathbf{T}(t_{j+1})$ and $\mathbf{N}(t_{j+1})$ for $\mathbf{x}_{(h)}(t)$. Define these to be $\mathbf{T}_{j+1}$ and $\mathbf{N}_{j+1}$, which amounts to

$$\begin{aligned}
\mathbf{T}_{j+1} &= \sin[\kappa(t_j)v(t_j)h]\mathbf{N}_j + \cos[\kappa(t_j)v(t_j)h]\mathbf{T}_j \\
\mathbf{N}_{j+1} &= \cos[\kappa(t_j)v(t_j)h]\mathbf{N}_j - \sin[\kappa(t_j)v(t_j)h]\mathbf{T}_j
\end{aligned}$$

Finally, replace j by $j+1$.

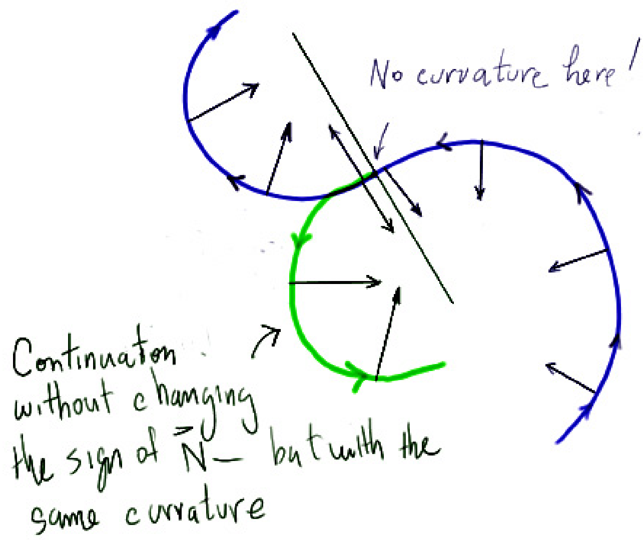
Notice that the algorithm breaks down for if for some t_j, $\kappa(t_j) = 0$: If this ever happened, the divisions in (1.39) and (1.40) would be undefined. When the curvature is zero, $\mathbf{N}$ is undefined, and this is what happens whenever one switches from turning left, say, to turning right. At the moment in between, one is going straight ahead, and $\mathbf{N}$ is undefined.

Here is a picture showing this:



The direction of travel along the curve is indicated by the arrows. The orthogonal vectors are the unit normal vectors at various points, indicating the direction of turning. At the point where the curve switches from turning left to turning right, **N** is undefined. It has a well defined one sided limits at this point though, and they are opposite to one another. Roughly speaking, "**N** changes sign at this point".

We can get another curve that has the exact same speed $v(t)$ and curvature $\kappa(t)$ by just continuing without changing the sign of **N** at this point. Here is a graph of it:



In so far as the graph is accurate, the second option for continuation of the curve past the point of zero curvature is just the reflection of the first one about the tangent line at that point.

You see from this example that whenever the curvature is zero for some values of t, there will be more than one curve with that curvature, even with the same speed. However, you also see what you need to do to uniquely specify the curve: You just need to specify whether **N** changes sign or not at the flat spot.

It is less crucial to the algorithm that the speed $v(t)$ also be strictly positive. However, if the speed were zero initially, the initial unit tangent vector would be undefined.

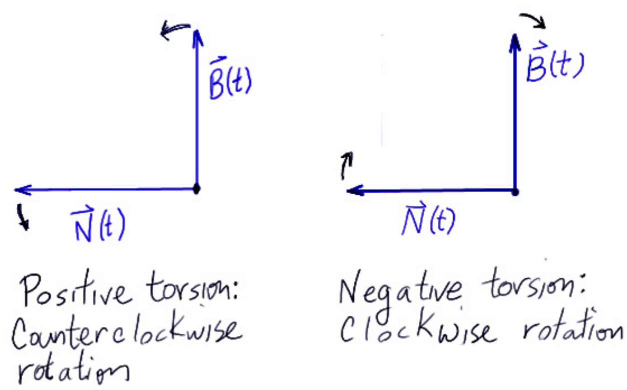
Now that we have understood the planar case, let us go on to the three dimensional case, and bring torsion into the picture. This will bring another step into our algorithm: After we have moved a step along the tangent circle, we must rotate the plane of motion, either clockwise or counterclockwise, according to the sign and magnitude of the torsion.

Let $\mathbf{T}_j$, $\mathbf{N}_j$ and $\mathbf{B}_j$ be the unit tangent, normal and binormal at time t_j. Let $\mathbf{T}_{j+1}$, $\tilde{\mathbf{N}}_{j+1}$ and $\tilde{\mathbf{B}}_{j+1}$ denote the unit tangent, normal and binormal after moving along the tangent circle a distance $v(t_j)h$, as before. We have put tildes over the normal and binormal as we are not done with them yet.

Now, to take the torsion into account, we make a rotation in the plane spanned by $\tilde{N}_{j+1}$ and $\tilde{B}_{j+1}$, which tilts the plane of motion. By one of the Frenet–Seret formulae,

$$B'(t) = v(t)\tau(t)\mathbf{N}(t) \ . \tag{1.41}$$

If v and τ are constant. Now consider the rotation depicted below:



Here

$$\begin{aligned} \mathbf{B}(t) &= \quad \cos(\theta(t))\mathbf{B}(0) + sin(\theta(t))\mathbf{N}(0) \\ \mathbf{N}(t) &= -\sin(\theta(t))\mathbf{B}(0) + \cos(\theta(t))\mathbf{N}(0) \end{aligned}$$

(Note: we are only taking into account the turning of $\mathbf{N}$ due to the torsion, and not the curvature. The latter happens in the plane of motion, which is not what we are graphing here.).

It follows that

$$\mathbf{B}'(0) = \theta - \theta'(0)\mathbf{N}(0) \ . \tag{1.42}$$

Comparing (1.41) and (1.42), we see that if v and τ are steady, then the total rotation over a time interval of length h is $\theta = v\tau h$.

Therefore, we take into account the torsion by updating $\tilde{N}_{j+1}$ and $\tilde{B}_{j+1}$ to produce N_{j+1} and B_{j+1} as follows:

$$\begin{aligned} \mathbf{B}_{j+1} &= \quad \cos(v(t_j)\tau(t_j)h)\tilde{\mathbf{B}}_{j+1} + sin(v(t_j)\tau(t_j)h)\tilde{\mathbf{N}}_{j+1} \\ \mathbf{N}_{j+1} &= -\sin(v(t_j)\tau(t_j)h)\tilde{\mathbf{B}}_{j+1} + \cos(v(t_j)\tau(t_j)h)\tilde{\mathbf{N}}_{j+1} \end{aligned}$$

Now, here is the curve reconstruction algorithm in three dimensions:

Curve contruction algorithm: We are given $\mathbf{x}(0)$, $\mathbf{T}(0)$ and $\mathbf{N}(0)$, as well as $v(t)$, $\kappa(t)$ and $\tau(t)$ for $0 \le t \le t_*$. We pick a (large) value of N, and define $h = t_*/N$. We assume that $\kappa(t) > 0$ for all $0 \le t \le t_*$.

(1) (initialization) Define $\mathbf{x}_{(h)}(0) = \mathbf{x}(0)$, and define $\mathbf{T}_0 = \mathbf{T}(0)$, $\mathbf{N}_0 = \mathbf{N}(0)$ and $\mathbf{B}_0 = bT_0 \times \mathbf{N}_0$.

Then, starting with $j = 0$, and **while** $j < N$:
(2) (segment contruction) Given $\mathbf{x}_{(h)}(t_j)$ and the orthonormal unit vectors $\mathbf{T}_j$ and $\mathbf{N}_j$, define the center $\mathbf{c}_j$ by

$$\mathbf{c}_j = \mathbf{x}_{(h)}(t_j) + (1/\kappa(t_j))\mathbf{N}_j \ , \tag{1.43}$$

and define $\mathbf{x}_{(h)}$ for $t_j \le t \le t_{j+1}$ by

$$\mathbf{x}_{(h)}(t) = \mathbf{c}_j + \frac{1}{\kappa(t_j)}\big(-\cos[\kappa(t_j)v(t_j)(t-t_j)]\mathbf{N}_j + \sin[\kappa(t_j)v(t_j)(t-t_j)]\mathbf{T}_j\big) \ . \tag{1.44}$$

(3) (updating $\mathbf{T}$*,* $\mathbf{N}$ *and* $\mathbf{B}$*, first part)* Using (1.40) compute $\mathbf{T}(t_{j+1})$ and $\mathbf{N}(t_{j+1})$ for $\mathbf{x}_{(h)}(t)$. Define these to be $\mathbf{T}_{j+1}$ and $\mathbf{N}_{j+1}$, while keeping $\mathbf{B}_j$ unchanged. This amounts to

$$\begin{aligned} \mathbf{T}_{j+1} &= \sin[\kappa(t_j)v(t_j)h]\mathbf{N}_j + \cos[\kappa(t_j)v(t_j)h]\mathbf{T}_j \\ \tilde{\mathbf{N}}_{j+1} &= \cos[\kappa(t_j)v(t_j)h]\mathbf{N}_j - \sin[\kappa(t_j)v(t_j)h]\mathbf{T}_j \\ \tilde{\mathbf{B}}_{j+1} &= \mathbf{B}_j \ . \end{aligned}$$

(4) (updating $\mathbf{T}$*,* $\mathbf{N}$ *and* $\mathbf{B}$*, second part)* Now tilt the plane of motion according to the torsion and the speed at time t_j:

$$\begin{aligned} \mathbf{B}_{j+1} &= \quad \cos(v(t_j)\tau(t_j)h)\tilde{\mathbf{B}}_{j+1} + sin(v(t_j)\tau(t_j)h)\tilde{\mathbf{N}}_{j+1} \\ \mathbf{N}_{j+1} &= -\sin(v(t_j)\tau(t_j)h)\tilde{\mathbf{B}}_{j+1} + \cos(v(t_j)\tau(t_j)h)\tilde{\mathbf{N}}_{j+1} \end{aligned}$$

Finally, replace j by $j+1$.

We now state a theorem telling us that if we take h small enough, we do get a good reconstruction of our curve.

Theorem 5 (Curve Reconstruction from curvature and speed in $I\!R^2$) *Let* $\mathbf{x}(t)$ *be a parameterized curve in* $I\!R^3$ *that is twice differentiable, and suppose that* $v(t) > 0$ *and* $\kappa(t) > 0$ *for all* t *in the time interval* $[0, t_*]$*. Let* $\mathbf{x}_{(h)}(t)$ *be the curve defined by the curve reconstruction algorithm above. Then for each* t *with* $0 \le t \le t_*$*,*

$$\lim_{t\to\infty} \mathbf{x}_{(h)}(t) = \mathbf{x}(t) \ .$$

We shall not prove the theorem here. The main point is that because the tangent circle approximation is a second order approximation, the error we make at each step in the recursion has a size proportional to h^2. We make $N = 1/h$ such mistakes, so in the end the cummulative error is proprtional to h, and hence vanished as h tends to zero.

1.7 The curvature and torsion are independent of the parameterization.

The same path can be parameterized many ways. For instance, in Examples 4 and 5, we considered two different parameterizations of the circle of radius r centered on the origin in $I\!R^2$. Though the *parameterized curves* in Examples 4 and 5 are different, and have different velocities and accelerations, they trace out the same path in the plane.

The difference between these two parameterizations is the rate at which the path is traced out. In Example 4, this is done at a steady rate, while in Example 5, the tracing is done faster and faster. It turns out, however, that the curvature at a point on the path is a purely geometric property of the path itself – it is independent of the parameterization. Not only that, so is the unit normal vector, and, up to a sign, so is the unit tangent vector.

To see this suppose that $\mathbf{x}(t)$ and $\mathbf{y}(u)$ are two parameterizations of the same path in $I\!R^n$. Suppose that

$$\mathbf{x}(t_0) = \mathbf{y}(u_0)$$

so that when $t = t_0$ and $u = u_0$, both curves pass through the same point. Let us "match up" the times for the two curves on intervals round t_0 and u_0 by defining the function $t(u)$ so that

$$\mathbf{y}(u) = \mathbf{x}(t(u)) \ . \tag{1.45}$$

That is, $t(u)$ is the time at which the first curve passes through $\mathbf{y}(u)$. As long as the parameterizations are one to one, which we shall suppose to be the case, the function $t(u)$ is well defined.

It turns out that if both prameterizations are differentiable, then $t(u)$ is differentiable as well. Let us assume that this is the case.

Then, by the chain rule and (1.45)

$$\mathbf{y}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t(u)) = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u)) \ .$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we get the following relation between the speed at which the two curve pass through the point in question:

$$|\mathbf{y}'(u_0)| = \left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| |\mathbf{x}'(t_0)| \ .$$

Therefore,

$$\begin{aligned} \frac{1}{|\mathbf{y}'(u_0)|}\mathbf{y}'(u_0) &= \left(\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right|^{-1}\frac{\mathrm{d}t}{\mathrm{d}u}\right)\frac{1}{|\mathbf{x}(t_0)|}\mathbf{x}'(t_0) \\ &= \pm\frac{1}{|\mathbf{x}(t_0)|}\mathbf{x}'(t_0) \ . \end{aligned}$$

The plus sign is correct is t is an increasing function of u, in which case the two parameterizations trace the path out in the same direction, and the minus sign is correct if t is a decreasing function of u.

This shows that up to a sign, the unit tangent vector $\mathbf{T}$ at the point in question comes out the same for the two parameterizaions.

Next, let us differentiate once more. We find

$$\begin{aligned} \mathbf{y}''(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}'(u) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u))\right) \\ &= \left(\frac{\mathrm{d}^2 t}{\mathrm{d}u^2}\right)\mathbf{x}'(t(u)) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''(t(u)) \ . \end{aligned}$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we find the following formula relating the acceleration along the two curves as they pass though the point in question:

$$\mathbf{y}''(u_0) = \left(\frac{\mathrm{d}^2 t}{\mathrm{d}u^2}\right)\mathbf{x}'(t_0) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''(t_0) \ .$$

Notice that the first term on the right is a multiple of $\mathbf{T}$, and hence when we decompose $\mathbf{y}''(u_0)$ into its tangential and orthogonal components, this piece contributes only to the tangential component. Hence

$$(\mathbf{y}''(u_0))_\perp = \left(\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 \mathbf{x}''(t_0)\right)_\perp = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 (\mathbf{x}''(t_0))_\perp \ .$$

Because of the square, $(\mathbf{y}''(u_0))_\perp$ is a positive multiple of $(\mathbf{x}''(t_0))_\perp$, and so these two vectors point in the exact same direction. That is,

$$\mathbf{N} = \frac{1}{|(\mathbf{y}''(u_0))_\perp|}(\mathbf{y}''(u_0))_\perp = \frac{1}{|(\mathbf{x}''(t_0))_\perp|}(\mathbf{x}''(t_0))_\perp \ ,$$

showing that the normal vector $\mathbf{N}$ is independent of the parameterization.

Next, we consider the curvature. Since

$$\begin{aligned} \frac{1}{|\mathbf{y}'(u_0)|^2}|(\mathbf{y}''(u_0))_\perp| &= \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^{-2} \frac{1}{|\mathbf{x}'(t_0)|^2}\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 |(\mathbf{x}''(t_0))_\perp| \\ &= \frac{1}{|\mathbf{x}'(t_0)|^2}|(\mathbf{x}''(t_0))_\perp| \ , \end{aligned}$$

we get the exact same value for the curvature at the same point, using either parameterization. This show that although in practice we use a particular parameterization to compute the curvature κ and the unit normal $\mathbf{N}$, the results do not depend on the choice of the paramerterization, and are in fact an intrinsically geometric property of the path that the curve traces out.

Let us go on to consider $\mathbf{B}(t)$ and $\tau(t)$. Since $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, it follows that $\mathbf{B}(t)$ is well defined, independent of the parameterization, up to a sign. Then, consideration of the formula

$$\mathbf{B}'(t) = v(t)\tau(t)\mathbf{N}(t)$$

under two parameterization shows that like the curvature, the torsion is independent of the parameterization. It is determined by the geometry of the track itself, and not how fast or slow we move along it.

1.8 The winding number of a plane curve

There is something special about plane curves; i.e., curve in $I\!R^2$. The special thing is that there is a well–defined notion of a counterclockwise rotation by $\pi/2$ in the plane. Given any vector $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the vector $\mathbf{v}^\perp$ given by

$$\mathbf{v}^\perp = \begin{bmatrix} -b \\ a \end{bmatrix}$$

is obtained from $\mathbf{v}$ by counterclockwise rotation through the angle $\pi/2$. Indeed, by the angle addition formulas,

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}^\perp = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{bmatrix} .$$

Therefore, for a planar curve, it makes sense to say whether the curve is turning in a clockwise or a counterclockwise direction: As we have seen, $\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$, so that $\mathbf{T}'(t)$ is either proportional to $(\mathbf{T}(t))^\perp$, or else to $-(\mathbf{T}(t))^\perp$. In the first case, the curve is "curving" counterclockwise, and in the second it is "curving" clockwise.

Intuitively, curvature has to do with turning. The more curvature a curve has, the more it turns round and round. Can we devise a formula for the cumulative "turning" that a plane curves makes in some given time interval?

Yes, and here is how: The vectors $\{\mathbf{T}(0), (\mathbf{T}(0))^\perp\}$ are a right handed orthonormal basis for $I\!R^2$. Taking this basis as our reference basis, we may identify these vectors with $\{\mathbf{e}_1, \mathbf{e}_2\}$. Now, at any later time, $\mathbf{T}(t)$ is a unit vector, and so it has the form

$$\mathbf{T}(t) = \begin{bmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{bmatrix} .$$

for some $\theta(t)$ that is defined up to a multiple of 2π.

In fact, we can define $\theta(t)$ *absolutely*; i.e., without an ambiguity about multiples of 2π. The key to this is to note that in any case, $\theta'(t)$ is well defined. Indeed,

$$\mathbf{T}'(t) = \theta'(t) \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

so that

$$\mathbf{T}(t)^{\perp} \cdot \mathbf{T}'(t) = \theta'(t) \ .$$

Defining $\theta(0) = 0$, we have

$$\theta(t) = \int_0^t \theta'(r)\mathrm{d}r = \int_0^t \left(\mathbf{T}^{\perp}(t) \cdot \mathbf{T}'(t)\right) \mathrm{d}r \ .$$

Notice that the right hand side is well defined for any twice differentiable curve. This formula give the cumulative angle of rotation of the unit tangent vector $\mathbf{T}$ up to time t.

Now consider a *closed curve* that periodically runs over the same track, over and over. That is, there is a time t_* so that for all t,

$$\mathbf{x}(t + t_*) = \mathbf{x}(t) \ .$$

The time t_* is called the *period* of the curve. It is easy to see that $\mathbf{x}(t_*) = \mathbf{x}(0)$, and also that

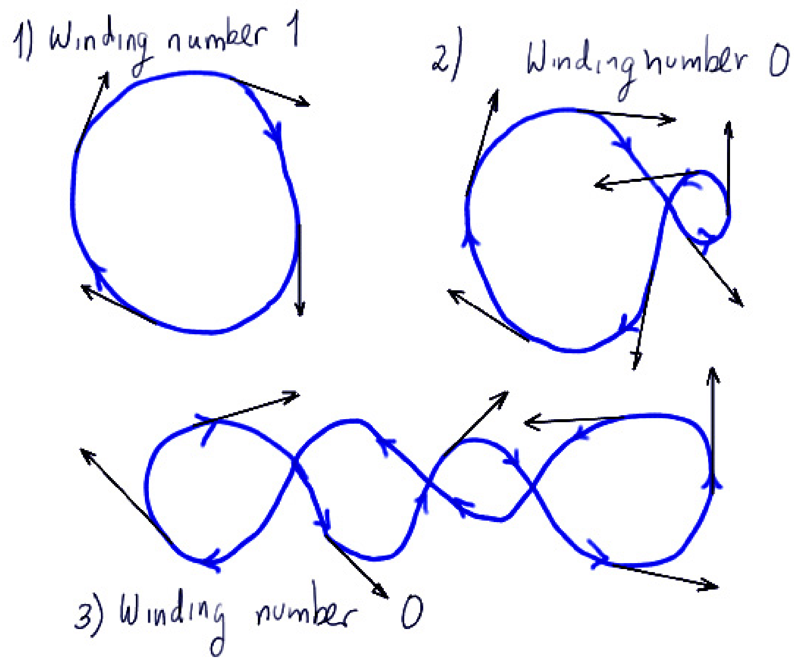
$$\mathbf{T}(t_*) = \mathbf{T}(0) \ .$$

Now, since $\mathbf{T}(t_*)$ points in the same direction as $\mathbf{T}(0)$, it must be that $\theta(t_*)$ is an integer multiple of 2π. This integer is called the winding number of the curve. This justifies the following definition:

Definition (Winding number of a planar) Let $\mathbf{x}(t)$ be a parameterized curve in the plane that is closed and periodic with period $t_* > 0$; i.e., $\mathbf{x}(t + t_*) = \mathbf{x}(t)$ for all t. Then the *winding number* of this curve it the integer

$$\frac{1}{2\pi} \int_0^{t_*} \left(\mathbf{T}^{\perp}(t) \cdot \mathbf{T}'(t)\right) \mathrm{d}r \ . \tag{1.46}$$

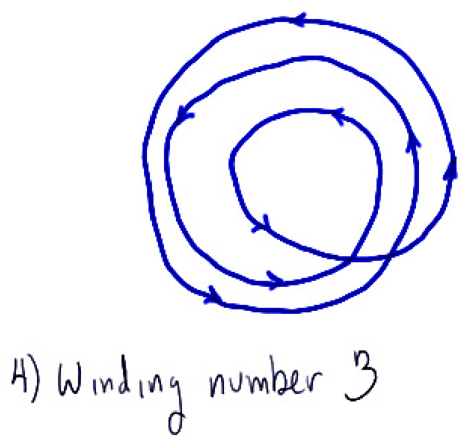
The winding number counts the number of times $\mathbf{T}$ turns around counterclockwise on one circuit of the curve. Here are some pictures of closed planar curves with the direction of motion indicated by arrows, and with some unit tangent vectors sketched in so that you can see the turning.



In the first curve, **T** turns around once counterclockwise, so the winding number is $+1$. If we reverse the direction of motion, it would be -1.

The second curve has two loops. one clockwise, and one counterclockwise, so the winding number is zero. The third curve has four loops. two clockwise, and two counterclockwise, so the winding number is zero here as well.

Next, here is a picture of a curve with winding number $+3$. We indicate the direction of motion by arrow on the curve, but do not sketch in unit tangent vectors this time to keep the graph from getting too busy.



We close by mentioning a more convenient formula for the winding number: First, it is easy to see that

$$\mathbf{T}^{\perp} = \frac{1}{v}\mathbf{v}^{\perp} \ ,$$

where we are suprresing the dependence on t in our notation to keep it simple.

Combing this with formula (1.19), namely $\mathbf{T}' = v\kappa\mathbf{N}$, we have

$$\mathbf{T}^\perp \cdot \mathbf{T}' = \kappa \mathbf{v}^\perp \cdot \mathbf{N} \ .$$

But since the component of $\mathbf{a}$ that is orthogonal to $\mathbf{v}$ is $\kappa v^2 \mathbf{N}$,

$$\kappa \mathbf{v}^\perp \cdot \mathbf{N} = \frac{1}{v^2} \mathbf{v}^\perp \cdot \mathbf{a} \ .$$

Combining results,

$$\mathbf{T}^\perp \cdot \mathbf{T}' = \frac{1}{v^2} \mathbf{v}^\perp \cdot \mathbf{a} \ ,$$

and so we have another formula that is equivalent to (1.46), but is more convenient for computation:

$$\frac{1}{2\pi} \int_0^{t_*} \left(\frac{1}{v^2(t)} \mathbf{v}^\perp(t) \cdot \mathbf{a}(t) \right) \mathrm{d}r \ . \tag{1.47}$$

Example 8 (Computing a winding number) Consider the planar curve

$$\mathbf{x}(t) = \begin{bmatrix} 1 + \cos(t) + \cos(2t) \\ \sin(t) + \sin(2t) \end{bmatrix} \ .$$

Notice that this curve is periodic with period $t_* = 2\pi$.

One easily computes

$$\mathbf{v}(t) = \begin{bmatrix} -\sin(t) - 2\sin(2t) \\ \cos(t) + 2\cos(2t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{a}(t) = -\begin{bmatrix} \cos(t) + 4\cos(2t) \\ \sin(t) + 4\sin(2t) \end{bmatrix} \ .$$

From here it easily follows that

$$v^2(t) = 5 + 4(\sin(t)\sin(2t) + \cos(t)\cos(2t)) \qquad \text{and} \qquad \mathbf{v}^\perp(t) = -\begin{bmatrix} \cos(t) + 2\cos(2t) \\ \sin(t) + 2\sin(2t) \end{bmatrix} \ ,$$

so that

$$\mathbf{v}^\perp(t) \cdot \mathbf{a}(t) = 9 + 6(\cos(t)\cos(2t) + \sin(t)\sin(2t) \ .$$

At this point is might look rather unlikely that (1.47) will yield an integer. But using the angle addition identities, one sees that actually

$$\sin(t)\sin(2t) + \cos(t)\cos(2t) = \cos(t) \ ,$$

and so the winding number is just

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{9 + 6\cos(t)}{5 + 4\cos(t)} \mathrm{d}t \ .$$

This intergal may now be evaluated, using the $tan(\theta/2$ substitution, for example, and one finds that the winding number is 2.

1.6 Speed and arc length

As we have explained above, the speed $v(t)$ represents the rate of change of the distance traveled with time. Given some reference time t_0, define

$$s(t) = \int_{t_0}^{t} v(u)\mathrm{d}u \ . \tag{1.48}$$

Then by the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = v(t)$$

and clearly $s(t_0) = 0$. Hence the rate of change of $s(t)$ is $v(t)$, which is the rate of change of the distance traveled with time, as one has moved along the path traced out by $\mathbf{x}(t)$.

Definition (Arc length) The function $s(t)$ defined by (1.48) is called the *arc length* along the path traced out by $\mathbf{x}(t)$ since time t_0.

Example 9 (Computation of arc length) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} t \\ 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as in the previous example. Then, as we have seen, for all $t > 0$, $v(t) = 1 + t$. Therefore,

$$s(t) = \int_0^t (1+u)\mathrm{d}u = t + \frac{t^2}{2} \ .$$

If you took a piece of string, and cut it so it can be run along the path from the starting point to the position at time t, the length of the string would be $t + (t^2/2)$ units of distance.

By definition, $v(t) \geq 0$, and so $s(t)$ has a non negative derivative. This means that it is an increasing function. As long as $v(t) > 0$; i.e., as long as the particle never comes to even an instantaneous rest, $s(t)$ is strictly monotone increasing.

Suppose also that $s(t)$ increases without bound, so that

$$\lim_{t\to\infty} s(t) = \infty \ .$$

Then for any $s \geq 0$, there is exactly one value of $t \geq 0$ so that

$$s(t) = s \ . \tag{1.49}$$

This value of t, considered as a function of s, is the inverse function to the arc length function:

$$t(s) = t \ . \tag{1.50}$$

It answers a very simple question, namely: *How much time will have gone by when the distance travelled is s units of length?*

If you can compute an explicit expression for $s(t)$, such as the result $s(t) = t + \frac{t^2}{2}$ that we found in Example 9, that one has to do to find the inverse function $t(s)$ is to solve (1.49) to find t in terms of s:

Example 10 (Time as a function of arc length) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} t \\ 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as in the previous example. Then, as we have seen, for all $t > 0$, $s(t) = t + (t^2/2)$, To find t as a function of s, write this as

$$s = t + \frac{t^2}{2}$$

and solve for t in terms of s. In this case,

$$t + \frac{t^2}{2} = \frac{1}{2}((t+1)^2 - 1)$$

so

$$t = \sqrt{2s+1} - 1 \ .$$

That is,

$$t(s) = \sqrt{2s+1} - 1 \ .$$

This function tells you how long it took to travel a given distance s when moving along the curve.

We can then get a new parameterization of our curve by defining $\mathbf{x}(s)$ by

$$\mathbf{x}(s) = \mathbf{x}(t(s)) \ .$$

This is called the *arc length parameterization.*

Example 11 (Converting to the arc length parameterization) Let $\mathbf{x}(t) = \begin{bmatrix} t \\ 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as in the previous example. Then, as we have seen, for all $t > 0$, $t(s) = \sqrt{2s+1} - 1$ Therefore,

$$\mathbf{x}(s) = \mathbf{x}(t(s)) = \begin{bmatrix} \sqrt{2s+1} - 1 \\ 2^{3/2}(\sqrt{2s+1} - 1)^{3/2}/3 \\ (\sqrt{2s+1} - 1)^2/2 \end{bmatrix} \ .$$

The arc length parameterization generally is complicated to work out explicitly. Even when you can work it out, it often looks a lot more complicated than whatever t parameterization you started with, as in the previous example. So why do we bother with it? What is it good for?

The point about the arc length parameterization is that it is purely geometric, so that it helps us to understand the geometry of a curve. If we compute the rate of change of the unit tangent vector $\mathbf{T}$ as a function of s, we are computing the rate of turning per unit distance along the curve. This is an intrinsic property of the curve itself. If we compute rate of change of the unit tangent vector $\mathbf{T}$ as a function of t, we are computing something that depends on how fast we are moving on the curve, and not just on the curve itself. Indeed, if we use the arc length parameterization, $v(s) = 1$ for all s, and so the factors involving speed drop out of all of our formulas. They simplify to, for example,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(s) = \mathbf{T}(s)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}(s) = \kappa(s)\mathbf{N}(s) \ .$$

Often, this last formula is taken as the definition of the normal vector $\mathbf{N}$ and curvature κ. It has the advantage that this definition is manifestly geometric, so that the normal vector $\mathbf{N}$ and curvature κ do not depend on the parameterization of the curve. It has the disadvantage that it is generally very difficult to explicitly work out the arc length parameterization. In order to more quickly arrive at computational examples, we have chosen the form of the definition that is convenient for computation.

We close this section by pointing out some direct formulas for computing the curvature κ based on Theorem 3. The formula (1.18); i.e.,

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v^2(t)\kappa(t)\mathbf{N}(t) \ ,$$

provides a simple means for computing the curvature without working out the arc length parameterization. The points is that the left hand side, $\mathbf{a}(t)$, is generally quite simple to compute – just take two derivatives. Once you have done this, you can extract the curvature. How you would do this depends on the dimension, though.

Suppose the curve is in $I\!R^3$. Then we can take the cross product of the velocity and the acceleration. Since $\mathbf{v}(t)$ is proportional to $\mathbf{T}(t)$ and since the cross product of parallel vectors is zero,

$$\begin{aligned}\mathbf{v}\times\mathbf{a} &= (v\mathbf{T})\times(v'\mathbf{T} + v^2\kappa\mathbf{N})\\ &= vv'\mathbf{T}\times\mathbf{T} + v^3\kappa\mathbf{T}\times\mathbf{N}\\ &v^3\kappa\mathbf{T}\times\mathbf{N} \ .\end{aligned}$$

Since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, $\mathbf{T}\times\mathbf{N}$ is another unit vector, and so

$$|\mathbf{v}\times\mathbf{a}| = v^3\kappa \ .$$

This gives us the formula

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} \ . \tag{1.51}$$

In two dimensions, we can simply take the cross product of $\mathbf{v}^\perp$ and $\mathbf{a}$ with the same effect:

$$\begin{aligned}\mathbf{v}^\perp\cdot\mathbf{a} &= (v\mathbf{T}^\perp)\cdot(v'\mathbf{T} + v^2\kappa\mathbf{N})\\ &= vv'\mathbf{T}^\perp\cdot\mathbf{T} + v^3\kappa\mathbf{T}^\perp\cdot\mathbf{N}\\ &= \pm v^3\kappa\mathbf{T}^\perp\cdot\mathbf{N}\end{aligned}$$

since $\mathbf{T}^\perp = \pm\mathbf{N}$ so that $\mathbf{T}^\perp\cdot\mathbf{N} = \pm 1$. This gives us the formula

$$\kappa = \frac{|\mathbf{v}^\perp\cdot\mathbf{a}|}{v^3} \ . \tag{1.52}$$

There can be no ambiguity about which formula holds in which dimension – the cross product is only defined in $I\!R^3$, and the "perping" operation is only defined in $I\!R^2$.

Example 12 (Computing the curvature) Let $\mathbf{x}(t) = \begin{bmatrix} t \\ 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as before. We have already computed $\mathbf{a}(t)$ and $\mathbf{v}(t)$. Taking their cross product, we find

$$\mathbf{v} \times \mathbf{a} = \frac{1}{2}\begin{bmatrix} \sqrt{2t} \\ -2 \\ \sqrt{2/t} \end{bmatrix} .$$

Since $v(t) = 1 + t$, we have

$$\kappa(t) = \frac{\sqrt{4 + 2t + 2/t}}{2(1+t)^3} .$$

Problems

1 Let $\mathbf{x}(t) = r\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ where $r > 0$. This is a parameterization of the unit circle.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute $s(t)$, the arclength at time t, and $t(s)$, the time to travel arc length s .

(d) Compute the normal and tangential components of the acceleration at time t.

(e) Compute $\kappa(t)$.

2 Let $\mathbf{x}(t) = \begin{bmatrix} t+1 \\ t^2 \end{bmatrix}$. This is a parameterization of the parabola $y = (x-1)^2$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute $s(t)$, the arclength at time t, and $t(s)$, the time to travel arc length s .

(d) Compute the normal and tangential components of the acceleration at time t.

(e) Compute $\kappa(t)$.

3 Let $\mathbf{x}(t) = \begin{bmatrix} t \\ 2\sqrt{t} \\ 1/t \end{bmatrix}$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Write down the indefinite integral that gives $s(t)$.

(d) Compute the normal and tangential components of the acceleration at time t.

(e) Compute $\kappa(t)$.

4 Let $\mathbf{x}(t) = \begin{bmatrix} \sqrt{t} \\ 2/\sqrt{t} \\ t \end{bmatrix}$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Write down the indefinite integral that gives $s(t)$.

(d) Compute the normal and tangential components of the acceleration at time t.

(e) Compute $\kappa(t)$.

5 Let $\mathbf{x}(t)$ be a twice differentiable curve in $I\!R^3$. Show that

$$\mathbf{B}(t) = \frac{\rho(t)}{v^3(t)}\mathbf{v}(t) \times \backslash(t) \ .$$

6 Let $\mathbf{x}(t)$ be a twice differentiable curve in $I\!R^3$. Show that the torsion $\tau(t)$ is given by

$$\tau(t) = \frac{1}{v(t)}\mathbf{N}(t) \cdot \mathbf{B}'(t) \ .$$

Then, using the result of Exercise 5, show that

$$\tau(t) = \frac{\rho^2(t)}{v^6(t)}\mathbf{v}(t) \times \mathbf{a}'(t) \cdot \mathbf{a}(t) \ .$$

7 For the curve in Exercise 3, compute the binormal vector $\mathbf{B}(t)$ and the torsion $\tau(t)$.

8 For the curve in Exercise 3, compute the binormal vector $\mathbf{B}(t)$ and the torsion $\tau(t)$.

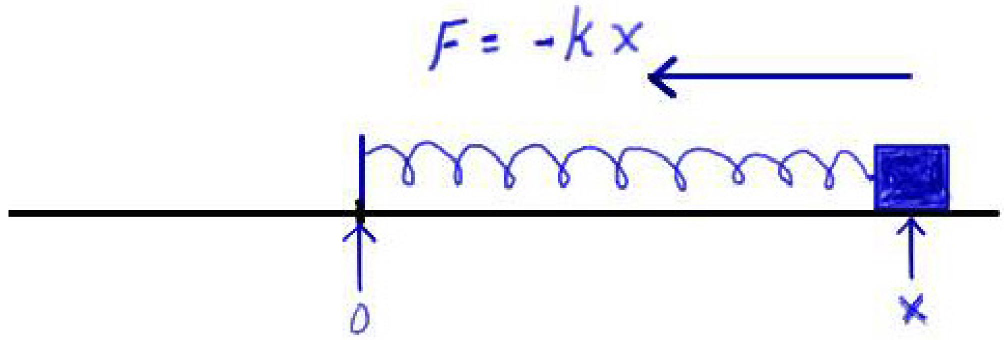
9 Let $\mathbf{x}(t) = \begin{bmatrix} 2\cos(5t) + \sin(t) \\ 2\sin(5t) - \cos(t) \end{bmatrix}$. Notice that this curve is periodic with period $t_* = 2\pi$. Compute its winding number.

10 Let $\mathbf{x}(t) = \begin{bmatrix} \cos(5t) + 4\sin(3t) \\ \sin(5t) - 4\cos(3t) \end{bmatrix}$. Notice that this curve is periodic with period $t_* = 2\pi$. Compute its winding number.

Section 2: Vector Fields and Motion

2.1 Differential equations

Consider a mass m attached to a spring. Identify 0 with the equilibrium position of the mass, and let x denote its displacement from equilibrium. If the spring is an ideal spring, obeying Hooke's law, there will be a force acting on the mass, pulling it back towards 0, and the strength of the force will be proportional to the displacement. If we let k denote the constant of proportionality, then the force is $F = -kx$.



Newton's second law says that the acceleration a of the mass, which is the second time derivative of the displacement; i.e., x'', is given by $F = ma$. This gives us the *differential equation*

$$-kx = mx'' \ .$$

Choosing units in which $k/m = 1$, and displaying the time dependence explicitly, we have

$$x''(t) = -x(t) \ . \tag{2.1}$$

This is a *second order equation*, meaning that it involves second derivatives.

If we are given the initial values of the position and velocity; i.e., $x(0)$ and $x'(0)$, there is exactly one function $x(t)$ satisfying (2.1). In other words,

• *A differential equation, like (2.1), together with appropriate initial data, is just a specification of a function.*

We are familiar with other ways to specify functions – in particular by formulas. We will soon see how to deduce that the only function $x(t)$ satisfying

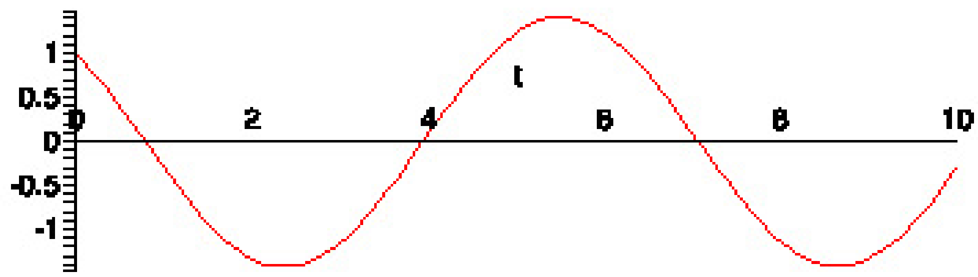
$$x''(t) = -x(t) \qquad \text{with} \qquad x(0) = x_0 \qquad \text{and} \qquad x'(0) = v_0 \tag{2.2}$$

is

$$x(t) = x_0 \cos(t) + v_0 \sin(t) \ . \tag{2.3}$$

You should be able to check that the function defined in (2.3) does satisfy everything imposed in (2.2). We will come back to the issue of uniqueness, but if there were other solutions, Newton's second law would have to be supplemented with something to select the "right" one. It does not have to be supplemented.

There is a third way we can specify a function: In terms of a graph. The graph is often the most useful specification, and one of the main reasons you might want to find an explicit formula such as (2.3) is that then you can graph x as a fucntion of t. For $x_0 = 1$ and $v_0 = -1$, and $0 \leq t \leq 10$, the graph would be:



What we are after here is an understanding of motion as described by a differential equation such as (2.1). Our goal will be to produce graphs that accurately describe the motion. If the differential equation involves second or other higher order derivatives, as does (2.1), the first step is to reduce the differential equation to a *system* of first order differential equations; i.e., differential equation involving only first derivatives.

2.2 Reduction to a system of first order differential equations

We begin the explanation with an example. Consider the second order equation (2.1). Introduce a new function $y(t)$ defined by $y(t) = x'(t)$. Then $y(t)$ is the velocity at time t, and clearly $y'(t) = x''(t) = -x(t)$. Hence we have the system of differential equations

$$\begin{aligned} x'(t) &= \quad y(t) \\ y'(t) &= -x(t) \end{aligned}$$

This is a *linear* system of first order differential equations. If we introduce the vector function $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We have

$$\mathbf{x}'(t) = A\mathbf{x}(t) \ .$$

We can express the initial conditions in (2.2) as $\mathbf{x}(0) = \mathbf{x}_0$ where $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$. The solution of the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \ . \tag{2.4}$$

is given by

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 \tag{2.5}$$

We know how to calculuate e^{tA} for any square matrix A. In this case, things are relatively easy because A is diagonalizable. As you can check, the eigenvalues are i and $-i$, and corresponding eigenvectors are $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$. Therefore, with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

$$\begin{aligned} e^tA &= U\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}U^* \\ &= \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} & i(e^{-it} - e^{it}) \\ i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

In the last equality, we have used Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \ .$$

You now see what the solution does: The initial data vector $\mathbf{x}_0$ just gets rotated at unit angular velocity in the counterclockwise direction.

Then, from (2.5),

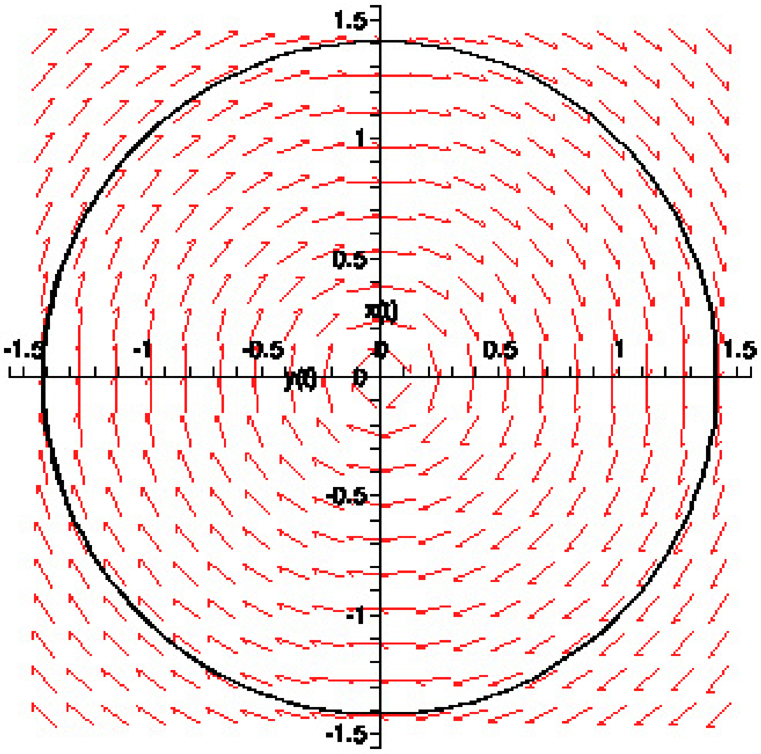
$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} x_0\cos(t) + v_0\sin(t) \\ v_0\cos(t) - x_0\sin(t) \end{bmatrix} \ .$$

In particular, we see that the x coordinate of $\mathbf{x}(t)$ is given by (2.3).

One reason for rewriting a second order equation as a system of first order equations is that when the system is linear, we can use (2.5) to solve it, as in the example we just discussed.

A second, and more important, reason is that it affords us a useful way to *visualize and think about* the motion described by the equation.

The visualization for (2.4) will be done in x, y plane, which is the space of all initial data for this eqaution. In this context, the x, y plane is often called the *phase plane* or *phase space* of the system. If we graph the solution curve $\mathbf{x}(t)$, we get what is called a *phase portrait* of the motion. Here is the phase portrait showing the solution of (2.4) for $0 \leq t \leq 10$.



The circle is the path traced out in the phase plane by the solution. If you proceed around the circle at constant speed, starting from $(1, -1)$, at unit angular velocity, the x–coordinate of your path traces out the sinusoidal graph pictured above in the first graph of this section.

Now, what about all of those little arrows in the phase portrait? They show the *direction and speed of motion* in the phase plane.

At each point $\begin{bmatrix} x \\ y \end{bmatrix}$, there is the vector $\begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. The corresponding unit vector, namely $\dfrac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x \\ y \end{bmatrix}$ gives the direction of motion through that point – and in the diagram, you see the path "following the arrows".

What we have here are two vector valued functions: For each input vector $\mathbf{x}$, there is an output vector that specifies the direction and speed of the motion though $\mathbf{x}$, or, in the second case, just the direction. Vector valued functions of this type are call *vector fields.* If you look at the picture above, you can see why. It almost looks some some sort of crop circle in a wheat field. Understanding vector fields in phase portraits is the key to understanding differential equations such as (2.3).

The ideas used to analyze this spring problem can be applied quite generally:

• *Faced with a second order differential equation – these come up all the time, through Newtons second law – we convert it to a system of first order equations by introducing a new velocity variable. We then try to solve this system. If it happens to be linear, we can do this by computing a matrix exponential.*

Example 1 (A damped spring) Again consider our spring, but this time, suppose that there is friction.

If the frictional force is proportional to the velocity, with proportionaliy constnat c – in units in which the $k/m = 1$ – we have that the force is $-x - cx'$. That is, we have the second order equation

$$x'' = -x - cx' \ .$$

Introducing the velocity $y = x'$, we have the first order system of differential equations

$$\begin{aligned} x'(t) &= \quad y(t) \\ y'(t) &= -x(t) - cx(t) \end{aligned}$$

This can be written in vector form as

$$\mathbf{x}' = A\mathbf{x}$$

where this time

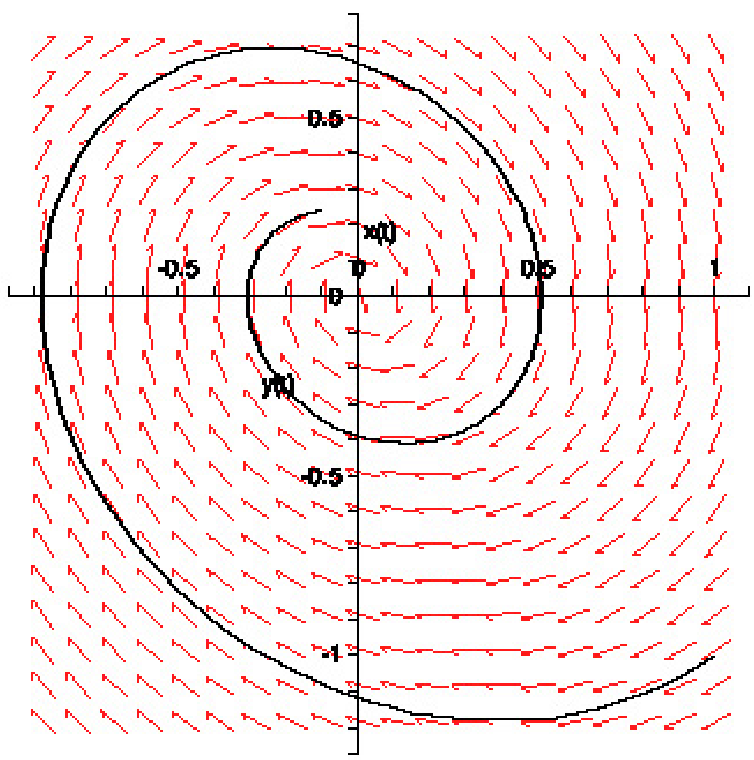
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \ .$$

Again, we have a linear system, and so the solution with $\mathbf{x}(0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.

So that we can draw graphs, we will choose a particular value of c. The value we choose is $c = 1/3$. Then, the result of an involved computation is that

$$e^{tA} = \frac{e^{-t/6}}{\sqrt{35}} \begin{bmatrix} \sqrt{35}\cos(t\sqrt{35}/6) + \sin(t\sqrt{35}/6) & 6\sin(t\sqrt{35}/6) \\ -6\sin(t\sqrt{35}/6) & \sqrt{35}\cos(t\sqrt{35}/6) + \sin(t\sqrt{35}/6) \end{bmatrix} \ ,$$

and the solution is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$. Here is the phase portrait:



The path spirals in this time, and you can see that this is what the vector field directs it to do. This corresponds with physical intuition: We expect the friction to "damp the motion out", so that the oscillations will become smaller and smaller. This is what you see in the phase portrait.

More generally, we might have the equation.

$$x''(t) = f(x'(t), x(t)) \ ,$$

where f is some given function of two variables. Then introducing $y(t) = x'(t)$, we have the equivalent system

$$\mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ f(y(t), x(t)) \end{bmatrix} \ .$$

This reduction method is very simple, and very useful. It can be applied much more generally – to equations of higher order, and even system if equations of higher order. This is indicated in the Exercises. In the next section we explain why the reduction is very useful, even when the resulting system is not linear.

2.3 Vector fields

The examples of systems of differential equations that we have discussed so far could be solved exactly because they were linear. In many other examples, the system will not be linear . That is, $\mathbf{x}'$ will not be a linear function of $\mathbf{x}$, and we cannot write our equation in the form $\mathbf{x}' = A\mathbf{x}$ for a constant matrix A. We will instead have $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ where the function $\mathbf{F}$ is a non linear vector field.

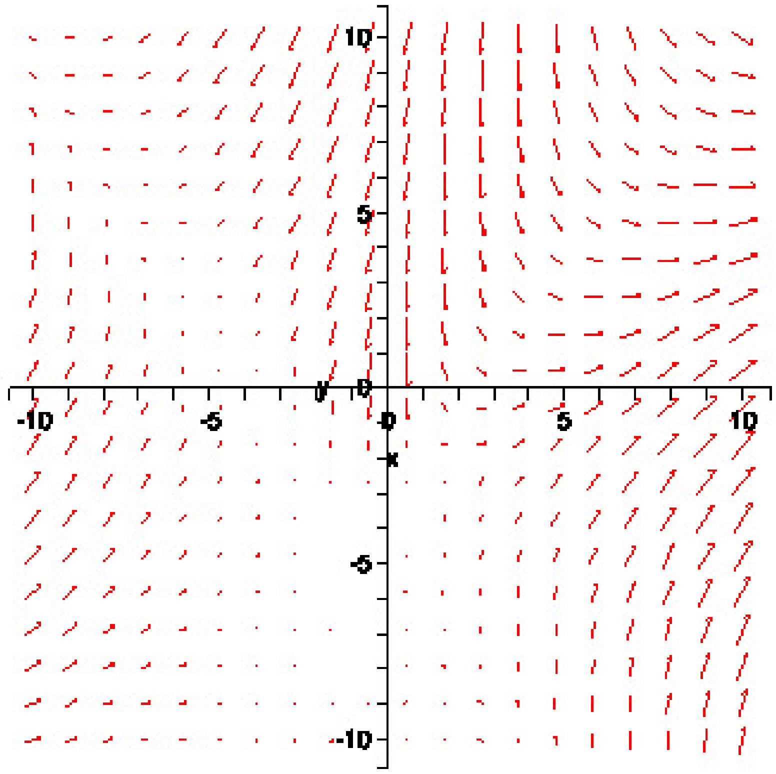
Definition (Vector field) A vector field on $I\!R^n$, or some open domain in $I\!R^n$, is simply a vector valued function $\mathbf{F}(\mathbf{x})$. That is, $\mathbf{F}$ is a function from $I\!R^n$ to $I\!R^m$. A vector field $\mathbf{F}$ is *linear* if and only if it can be written in the form $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ for some constant matrix A. Otherwise, it is *non linear*.

We will *usually* be interested in the case when $m = n$, and in particular, we will now focus on the case $m = n = 2$. In this case we can draw pictures.

For example, here is a non linear vector field:

$$\mathbf{F}(x, y) = (1 + x^2 + y^2)^{-1/2} \begin{bmatrix} (y + x + 4)(3x - y - 1) \\ (y + 2x + 8)(2x - 2y - 7) \end{bmatrix} \ . \qquad (2.6)$$

This is very non linear. Here is a plot of this vector field. with a “field” of arrows drawn at the points of a grid.



Think of the arrows as direction and speed indicators attached to a dense grid of sign-posts. These arrows tell you which direction you should go, and how fast. The speed is indicated by the length of the vector.

From this point of view, a vector field describes a "flow" across the phase plane. You just move across the plane, following the arrows at every step.

In other words, we can think of a vector field as giving a set of *instructions for motion* in the x, y plane. If you choose a starting point in the plane, and then move according to the instructions encoded into the vector field, you will trace out a curve. Such a curve is called a *flow line* of the vector field. These curves are the graphs of the solutions of the system of differential equations, and you see one in each of the phase portraits we drew above. The question before us is:

• *How do we read out the "instructions for motion" coded into the vector field, and graph the flow lines?*

2.4 The Euler Scheme

There is a very simple and straight forward algorithm for finding the flow lines – or at least a good approximation to them. If our goal is to draw a graph, we do not really need the *exact* curves anyhow: Any error of approximation that shifts the curve less than one pixel will be invisible anyway. This simple algorithm is known as the *Euler scheme*, or sometimes the *forward Euler scheme*. It builds the path up "step by step" in a "connect the dots" fashion".

The Euler Scheme

Fix a time step $h > 0$, and a starting point $\mathbf{x}_0$. Recursively define a sequence of points $\{\mathbf{x}_n\}$, $n \geq 0$ by

$$\mathbf{x}_n = \mathbf{F}(\mathbf{x}_{n-1})h \qquad \text{for all} \quad n \geq 1 \ . \tag{2.7}$$

Define a vector valued function $\mathbf{x}_{(h)}(t)$ by "connecting the dots": For any $t > 0$, let k be the largest integer such that $kh \leq t$. Then put

$$\mathbf{x}_{(h)}(t) = \mathbf{x}_k + (t - kh)\mathbf{x}_{k+1} \ . \tag{2.8}$$

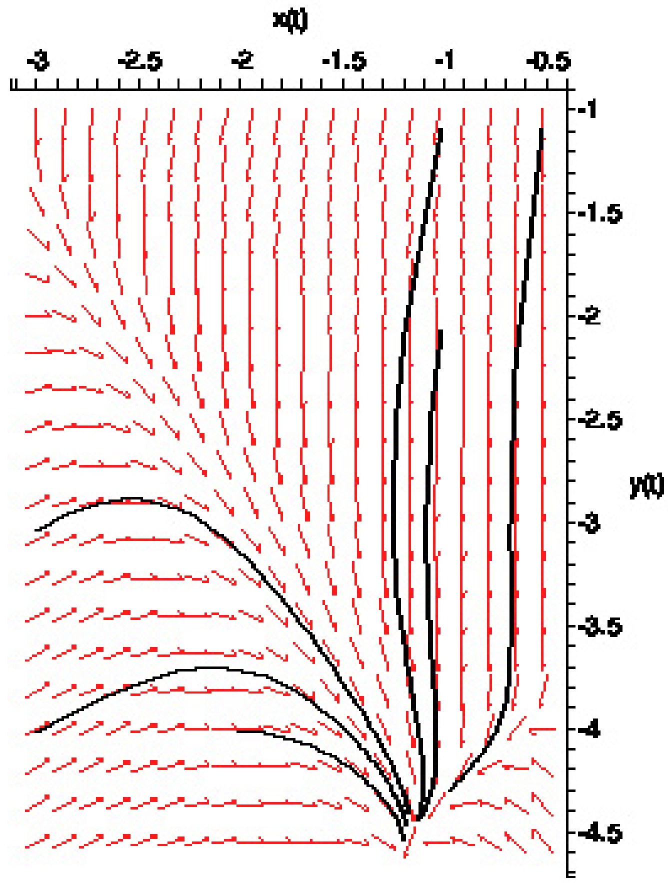
The Euler scheme involves the choice of the time step h. How to choose h is a matter that requires some thought. But for now, let's bypass all that. Once h is chosen, the Euler scheme is very easy to implement.

The terms of the recursively defined sequence are:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 + \mathbf{F}(\mathbf{x}_0)h \\ \mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{F}(\mathbf{x}_1)h \\ \mathbf{x}_3 &= \mathbf{x}_2 + \mathbf{F}(\mathbf{x}_2)h \\ \vdots &= \qquad \vdots \end{aligned} \tag{2.9}$$

The idea is that when you are at $\mathbf{x}_0$, you should move in the direction $(1/|\mathbf{F}(\mathbf{x}_0)|)\mathbf{F}(\mathbf{x}_0)$ with the speed $|\mathbf{F}(\mathbf{x}_0)|$. In h units of time, this would carry you to $\mathbf{x}_0 + \mathbf{F}(\mathbf{x}_0)h$, which is what we have called $\mathbf{x}_1$. From there, you move in the direction $(1/|\mathbf{F}(\mathbf{x}_1)|)\mathbf{F}(\mathbf{x}_1)$ with the speed $|\mathbf{F}(\mathbf{x}_1)|$. In h units of time, this would carry you to $\mathbf{x}_1 + \mathbf{F}(\mathbf{x}_1)h$, which is what we have called $\mathbf{x}_2$. Connecting up the dots, we get a path in the phase plane. It is certainly intuitively clear that if we choose a small value of h, and implement the Euler scheme, we will not be doing a bad job of "following the instructions for motion" encoded into the vector field.

For example, consider the vector field $\mathbf{F}$ given by (2.6). We consider six different starting points $\mathbf{x}_0$: $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$ $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$. Using the time step the time step $h = 0.05$, and running 60 steps so that the final time corresponds to $t = 3$, we find the paths:



What you see are the graphs of $\mathbf{x}_{(0.05)}$ for eight different starting points. The idea, which turns out to be correct for reasonable vector fields like this one, is that when h is small enough, $\mathbf{x}_{(h)}(t)$ is very close to the true solution $\mathbf{x}(t)$. More precisely, we can hope that

$$\lim_{h\to 0} \mathbf{x}_{(h)}(t)$$

exists, and if we let $\mathbf{x}(t)$ denote the limit, then $\mathbf{x}(t)$ is a solution to $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ with $\mathbf{x}(0) = \mathbf{x}_0$, and that there is no other solution. We will come back to this point later. For now, look at the graph, and notice how the flow lines "follow the directions" given by the arrows.* Showing the flow lines of a vector field is a good way to present it visually, perhaps better than as a field of arrows.

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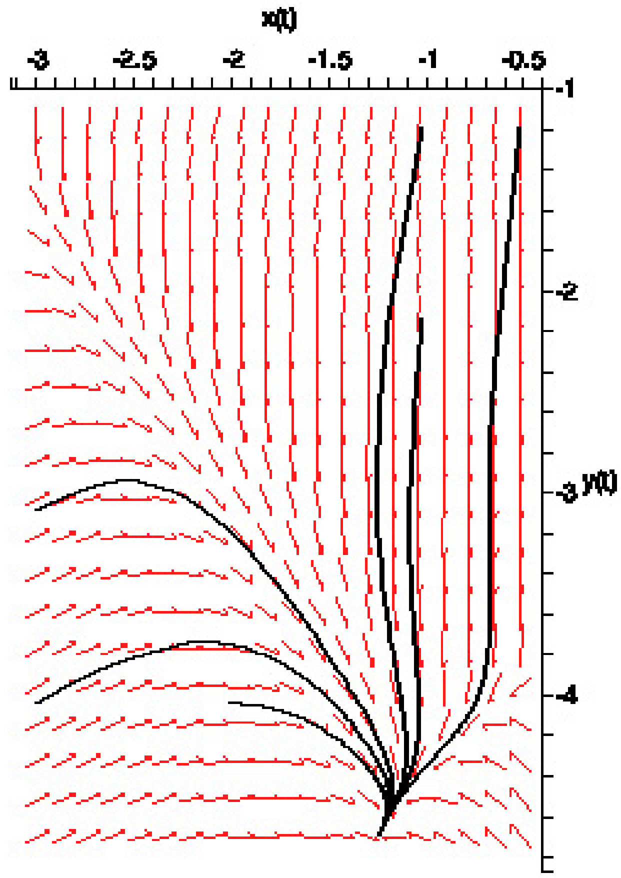
* The graph was generated in Maple. There are some serious problems with the way Maple draws the vector fields in. The paths, or "flow lines" are supposed to be following the arrows, and so the arrows should be tangent to the flow lines. This is more or less the case in the lower part of the graph, but not in the upper right part. What is going on?

It seems that Maple "rounds off" the angles of the arrows, Because of this rounding, the the horizontal component of the vector field is getting rounded off to zero in the upper right of the graph. It *looks* like the vector field pointing straight down in the upper right portion. This is not quite the case, and by looking at the flow lines – which are computed without inappropriate rounding, we get a better idea of what directions for motion are actually encoded into the vector field.

Notice that it looks like the flow lines are all converging on a single point. Indeed, if we run the algorithm for a longer time, taking enough steps to reach $t = 8$, we see:



The paths really do seem to be converging to a single point independent of the starting point, at least in the region in which we choose our starting points. This is an interesting phenomenon. But we have already covered a great deal of ground. Let's look back over what we have learned, and try to clarify our objectives in studying this subject.

2.5: What goals should we set?

When faced with a system of differential equations, it is generally not possible to find a formula for the solution. We did that in the last subsection when the system was linear; i.e., when the vector field $\mathbf{F}(\mathbf{x})$ was of the form $A\mathbf{x}$ for a constant matrix A.

For the system (2.6), this is just not possible. Just as certain integrals like $\int_0^x e^{-t^2/2}\mathrm{d}x$, cannot be evaluated in terms of the elementary functions, neither can most solutions of non linear differential equations. Instead they define new functions, just as $\int_0^x e^{-t^2/2}\mathrm{d}x$ defines the *error function* $\mathrm{erf}(x)$.

Many of the so–called *special functions* arise this way: They are the solutions of differential equations that "come up" all of the time when solving all kinds of problems. You may have heard of, say, Bessel functions of various kinds. They come from solving such equations.

If our goal is not going to be computing explicit formulas, what then will it be? There *are* things we *can* compute that tell us important information about the pattern of motion described by a vector field.

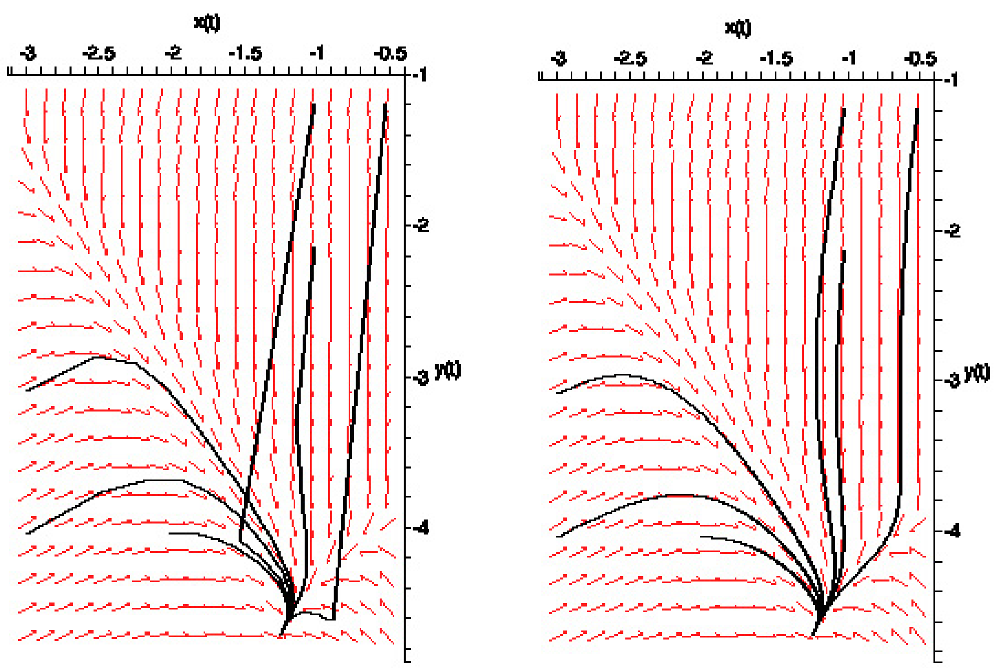
For example, looking at the last two phase portraits, you see that for the vector field (2.6), a wide variety of starting points get "swept in" towrds a point near $(-1.5, -4.75)$, and then they stop there. This is an example of a *stable equilibrium point.* We will see in the next section how to calculate them exactly. This one turns out to be exactly $(-5/4, -19/4)$. We can then give a rather precise description of the motion for staring points near this equilibrium point: *The flow line quickly approaches the equilibrium point* $(-5/4, -19/4)$*, and comes to rest there.* The point we wish to make is this:

• *While vector fields can describe quite complicated motions for which there are no simple formulas, the long time behavior of the flow lines can be quite simple indeed, and this is something we can calculate.*

In the next section, we will see how to do the calculations that locate any "equilibrium points" relevant to the long time behavior of the solutions.

If you really need to know further details of the motion – for instance, if you want to see the actual shape of a particular flow line over a short time – then you must in general resort to the Euler scheme, or something like it.

There are various refinements Euler scheme, but they all have one thing in common with the original: They all involve the choice of a time step $h > 0$ The following graphs show the paths produced by (2.9) with $h = 0.15$ on the left and $h = 0.01$ on the right, and $0 \leq t \leq 8$. Recall that before we used $h = 0.05$.



The one on the left with $h = 0, 15$ is rather differerent from what we found with $h = 0, 5$. On the other hand, the one on the right with $h = 0.01$ is pretty much indistinguishable from the one with $h = 0.05$.

The smaller h is, the better the approximation is. The point is that $\mathbf{F}$ is in general not constant as one moves along the straight line segment joining $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$. By the time one has gotten to $\mathbf{x}_{n+1}$, the vector field has changed from $\mathbf{F}(\mathbf{x}_n)$ to $\mathbf{F}(\mathbf{x}_{n+1})$. Since you are using $\mathbf{F}(\mathbf{x}_n)$ as your velocity all the way from $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$, ignoring the change in the vector field along the way, you are not *really* following the intstructions encoded in the vector field.

However, if h is small, then $\mathbf{x}_{n+1}$ will be close to $\mathbf{x}_n$. Hence, if $\mathbf{F}$ is nicely continuous, it will be the case that $\mathbf{F}(\mathbf{x}_n) \approx \mathbf{F}(\mathbf{x}_{n+1})$. To the extent that this is true, it doesn't matter that we were using the wrong velocity: While it was not *exactly* right, it was not *too far off*, and the difference would not show up in a graph unless it we drawn on a very, very fine scale. Hence if h is small enough, we can expect the Euler scheme to give us an accurate phase portrait.

However, we do not want to take h any smaller than necessary to get the shape of the flow lines right. The point is, that if you are trying to draw the flow lines for $0 \le t \le 1$, say, and you choose $h = 10^{-6}$, it will take 10^6 steps to reach $t = 1$. Each step involves computation, and if we choose h too small, we are wasting a great deal of computational effort.

There is a balancing act to be performed here. We need to choose h small enough to get the shape right, but no more, so that we do not waste computational effort. The key question then is this:

• *For a given vector field $\mathbf{F}$, a given starting point $\mathbf{x}_0$, and a given final time t, how small does h have to be for the Euler scheme to produce an accurate phase portrait?*

The fact that there is always some value $h > 0$ that is small enough is contained in the following theorem, which we state, but do not prove.

Theorem 1 (Existence of flow lines) *Let $\mathbf{x}_0$ be given, and let Ω be an open set containing $\mathbf{x}_0$. Let R be the distance from $\mathbf{x}_0$ to the exterior of Ω. Let $\mathbf{F}$ be a vector filed defined on Ω, Suppose that there are numbers B and C so that for all $\mathbf{x}$ in Ω,*

$$|\mathbf{F}(\mathbf{x})| \le B \qquad \text{and} \qquad \|J_{\mathbf{F}}(\mathbf{x})\| \le C \ .$$

Then there is a unique solution to

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.10}$$

that is defined until $\mathbf{x}(t)$ reaches the boundary of Ω if it ever does. In any case, this does not happen for $t < R/B$, so $\mathbf{x}(t)$ is defined for all such t. Moreover, the approximate solution $\mathbf{x}_{(h)}$ produced by the Euler scheme converges to the exact solution:

$$\lim_{h \to 0} \mathbf{x}_{(h)}(t) = \mathbf{x}(t)$$

for all times t for which $\mathbf{x}(t)$ is defined.

Exercises

1. Consider the second order differential equation

$$x''(t) = -x^2(t) + (x^2(t))' - 2x(t) \ .$$

Find a vector field $\mathbf{F}(\mathbf{x})$ so that this equation is equivalent to the first order system $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

2. Consider the second order differential equation

$$x''(t) = x^2(t) - (x'(t))^3 - 3x(t)x'(t) \ .$$

Find a vector field $\mathbf{F}(\mathbf{x})$ so that this equation is equivalent to the first order system $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

3. Bessel's equation is

$$x''(t) = \frac{(m^2 - t^2)x(t) = tx'(t)}{t^2} \ ,$$

where m is a fixed number. The right hand side explicitly involves t, and so you have to use a *time dependent* vector field $\mathbf{F}(\mathbf{x}, t)$ to write this equation as a system. Such systems with a time dependent vector field are called *non autonomous.* Find a time dependent vector field $\mathbf{F}(\mathbf{x}, t)$ so that this equation is equivalent to the first order system $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

4. Find the exact solution of $\mathbf{x}'(t) = \mathbf{F}(x(t))$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$.

5. Find the exact solution of $\mathbf{x}'(t) = \mathbf{F}(x(t))$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

6. Find the exact solution of $\mathbf{x}'(t) = \mathbf{F}(x(t))$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

7. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} (x-y)(1-x^2) \\ -y^3 - x \end{bmatrix} \ .$$

Take $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $h = 0.1$. Compute the points $\mathbf{x}_n$ generated by the Euler scheme for $n \leq 10$, and plot an approximate phase portrait based on your computations. Show the vector field $\mathbf{F}$ at each of the points $\mathbf{x}_n$ and at some nearby points.

8. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} (x^2-y)(1-x) \\ 3x - y \end{bmatrix} \ .$$

Take $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $h = 0.1$. Compute the points $\mathbf{x}_n$ generated by the Euler scheme for $n \leq 10$, and plot an approximate phase portrait based on your computations. Show the vector field $\mathbf{F}$ at each of the points $\mathbf{x}_n$ and at some nearby points.

9. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} y \\ -x \end{bmatrix} \ .$$

Take $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $h = 0.2$. The exact solution traces out a circle of radius 5 in the x, y plane.

(a) Compute the points $\mathbf{x}_n$ generated by the Euler scheme for $n \leq 20$, and plot an approximate phase portrait based on your computations, and plot an approximate phase portrait based on your computations. Show the vector field $\mathbf{F}$ at each of the points $\mathbf{x}_n$. The approximation that you get will not "close" the circle. Instead, it will "spiral out".

(b) To understand the spiraling, consider the vector field, but an arbitrary $h > 0$. Show that if $\{\mathbf{x}_n\}$ is generated using the Euler scheme, then for each n, $\mathbf{F}(x_n)$ is orthogonal to $\mathbf{x}_n$, and the Pythagorean Theorem yields

$$|\mathbf{x}_{n+1}|^2 = (1 + h^2)|\mathbf{x}_n|^2 \ .$$

Hence $\{\mathbf{x}_n\}$ is an increasing sequence.

(c) Let N be the smallest integer with $Nh > 2\pi$ Then N steps would be required to "close" the circle. By what we have seen above, it won't exactly close. But if in our graph the difference between $|\mathbf{x}_N|$ and $|\mathbf{x}_0| = 5$ is less than about half a pixel width, the flaw will be invisible. If a pixel witdth is 0.05 distance units, how small must h be so that the path will appear to close in a circle?

Section 3: Long time Behavior

3.1 Equilibrium points for systems of differential equations

Consider a starting point $\mathbf{x}_0$ for which $\mathbf{F}(\mathbf{x}_0) = 0$. Then if we define $\mathbf{x}(t) = \mathbf{x}_0$ for all t, we have $\mathbf{x}'(t) = 0$ for all t, and so $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) = \mathbf{F}(\mathbf{x}_0) = 0$ for all t. In other words, we have solved the differential equation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ for this initial data. The solution is very simple: There is no motion at all; the solution remains "in equilibrium" at $\mathbf{x}_0$.

Definition Let $\mathbf{F}$ be a vector field. Any point $\mathbf{x}$ such that $\mathbf{F}(\mathbf{x}) = 0$ is an *equilibrium point* for the vector field, and the corresponding flow.

Finding equilibrium points is then just a matter of solving the equation $\mathbf{F}(\mathbf{x}) = 0$. We have effective means for doing this – for example, we could use Newton's method. The examples that follow here have been chosen so that $\mathbf{F}(\mathbf{x}) = 0$ can be solved by simple algebraic means. This is slightly artificial, but it lets us focus on the new ideas.

Example 1 (Finding Equilibrium Points) Consider the vector field

$$\mathbf{F}(x,y) = (1 + x^2 + y^2)^{-1/2} \begin{bmatrix} (y + x + 4)(3x - y - 1) \\ (y + 2x + 8)(2x - 2y - 7) \end{bmatrix} \tag{3.1}$$

that was introduced in (2.6) and graphed there. If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = 0$ then

$$(y + x + 4)(3x - y - 1) = 0 \qquad \text{and} \qquad (y + 2x + 8)(2x - 2y - 7) = 0\ .$$

This is the case if an only if at least one of the factors in each product is zero. Therefore if $y + x + 4$, then we also must have either $y + 2x + 8 = 0$ or $2x - 2y - 7 = 0$. That is, either

$$\begin{aligned} x + y &= -4 \\ 2x + y &= -8 \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} x + y &= -4 \\ 2x - 2y &= 7 \end{aligned} \tag{3.3}$$

Likewise, if $3x - y - 1 = 0$, then x, y must solve either

$$\begin{aligned} 3x - y &= 1 \\ 2x + y &= -8 \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} 3x - y &= 1 \\ 2x - 2y &= 7 \end{aligned}\ . \tag{3.5}$$

solving these four linear systems, we find all equilibrium points of $\mathbf{F}$.

You easily find that the solution of (3.2) is $x = -4, y = 0$, that the solution of (3.3) is $x = -1/4, y = -15/4$, that the solution of (3.4) is $x = --7/5, y = -26/5$, and finally, that the solution of (3.5) is $x = -5/4, y = -19/4$ These are the equilibrium points for this vector field.

Let $\mathbf{F}$ be the vector field from Example 1, and let $\mathbf{x}_0 = -\dfrac{1}{4}\begin{bmatrix} 5 \\ 19 \end{bmatrix}$ be the fourth equilibrium point found there. If you look back at the graphs of flow lines for this vector field that were drawn in the previous section, you see that this appears to be the point of convergence of the flow lines we drew there. What is going on at this equilibrium point that makes it "pull in" the nearby flow lines?

3.2 Linearization of a system differential equations

It turns out that a system of differential equations can be "simplified" near an equilibrium point. The simplified system is linear, and therefore explicitly solvable. By looking at the explicit solution of the simplified system – that is, linearized system – we can determine the pattern of flow lines near the equilibrium point.

To see how this works, let's consider $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ where $\mathbf{F}$ is given by (3.1), and the starting point is $\mathbf{x}_0 + \mathbf{z}_0$. Here $\mathbf{x}_0$ is our equilibrium point, and $\mathbf{z}_0$ is a small "shift" away from it. This new point is not an equilibrium point when $\mathbf{z}_0$ is small. (The other three equilibria are not very close).

Let $\mathbf{x}(t)$ denote the solution of

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 + \mathbf{z}_0\ , \tag{3.6}$$

and define $\mathbf{z}(t)$ by

$$\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{x}_0\ . \tag{3.7}$$

That is, $\mathbf{z}(t)$ gives the displacement at time t from the equilibrium position $\mathbf{x}_0$. How does this change with time?

Since $\mathbf{x}_0$ is constant,

$$\mathbf{z}'(t) = (\mathbf{x}(t) - \mathbf{x}_0)' = \mathbf{x}'(t)\ .$$

From (3.7), $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{z}(t)$. Combining this with (3.7) and (3.6), we have

$$\begin{aligned} \mathbf{z}'(t) = \mathbf{x}'(t) &= \mathbf{F}(\mathbf{x}(t)) \\ &= \mathbf{F}(\mathbf{x}_0 + \mathbf{z}(t))\ . \end{aligned} \tag{3.8}$$

Now making a first order Taylor approximation,

$$\mathbf{F}(\mathbf{x}_0 + \mathbf{z}(t)) \approx \mathbf{F}(\mathbf{x}_0) + [J_{\mathbf{F}}(\mathbf{x}_0)]\mathbf{z}(t)\ . \tag{3.9}$$

Now, *since* $\mathbf{x}_0$ *is an equilibrium point*, $\mathbf{F}(\mathbf{x}_0) = 0$, and this simplifies to

$$\mathbf{F}(\mathbf{x}_0 + \mathbf{z}(t)) \approx [J_{\mathbf{F}}(\mathbf{x}_0)]\mathbf{z}(t)\ . \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\mathbf{z}'(t) \approx [J_{\mathbf{F}}(\mathbf{x}_0)]\mathbf{z}(t)\ . \tag{3.11}$$

Making this approximation, and remembering the initial condition $\mathbf{z}(0) = \mathbf{z}_0$, we obtain the linear system

$$\mathbf{z}'(t) = [J_{\mathbf{F}}(\mathbf{x}_0)]\mathbf{z}(t) \qquad \text{with} \qquad \mathbf{z}(0) = \mathbf{z}_0 \ . \tag{3.12}$$

Definition Let $\mathbf{x}_0$ be an equilibrium point for the differential equation $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$. Suppose that $\mathbf{F}$ is differentiable at $\mathbf{x}_0$. Then the *linearization of* $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$ *near* $\mathbf{x}_0$ is the linear system

$$\mathbf{z}'(t) = [J_{\mathbf{F}}(\mathbf{x}_0)]\mathbf{z}(t) \ .$$

To keep the notation simple, let A denote the matrix Jacobian matrix $J_{\mathbf{F}}(\mathbf{x}_0)$. Then the linearized initial value problem (3.12) is just

$$\mathbf{z}'(t) = A\mathbf{z}(t) \qquad \text{with} \qquad \mathbf{z}(0) = \mathbf{z}_0 \tag{3.13}$$

which has the solution

$$\mathbf{z}(t) = e^{tA}\mathbf{z}_0 \ . \tag{3.14}$$

Now, because of the approximations we have made, the curve $\mathbf{z}(t)$ in (3.14) is not the same as the one in (3.7). *However*, as long a $|\mathbf{z}(t)|$ stays small, so that the approximation in (3.9) is reasonably accurate, these two curves will be approximately equal. Therefore, using the one from (3.14) in (3.7), we get

$$\mathbf{x}(t) \approx \mathbf{x}_0 + e^{tA}\mathbf{z}_0 \ . \tag{3.15}$$

This approximation remains good as long as $|\mathbf{z}(t)| = |e^{tA}\mathbf{z}_0|$ remains small.

• *Therefore, at least in a small neighborhood of an equilibrium point, we can effectively graph the flow lines of the non linear system* $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$ *by graphing the flow lines of its linearization at* $\mathbf{x}_0$*. This is easy because the flow lines if a linear system can be computed exactly, using the matrix exponential.*

Example 2 (Computing a linearization) Let $\mathbf{F}$ be the vector field from Example 1, and let $\mathbf{x}_0$ be the equilibrium point found there by solving (3.5); i.e., $\mathbf{x}_0 = -\frac{1}{4}\begin{bmatrix} 5 \\ 19 \end{bmatrix}$

Next, we work out $J_{\mathbf{F}}(\mathbf{x}_0)$. Doing the computations, we find

$$J_{\mathbf{F}}(\mathbf{x}_0) = \frac{\sqrt{8}}{\sqrt{201}}\begin{bmatrix} -6 & 2 \\ 3/2 & -3/2 \end{bmatrix} \ .$$

Hence we see that the solution of (3.12) is given by

$$\mathbf{z}(t) = e^{tJ_{\mathbf{F}}(\mathbf{x}_0)}\mathbf{z}_0 = e^{tA}\mathbf{z}_0$$

where

$$A = \sqrt{8/201}\begin{bmatrix} -6 & 2 \\ 3/2 & -3/2 \end{bmatrix} \ . \tag{3.16}$$

The linearized system in Example 2 may look ugly on account of the square roots. However, there is something very, very nice about the matrix A in (3.16): *Both of its eigenvalues are negative.* A simple computation shows that the eigenvalues are

$$-\sqrt{2/201}(15 \pm \sqrt{129}) \ .$$

Since these are distinct, the matrix A can be diagonalized. That is, for some 2×2 matrix V,

$$A = V \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} V^{-1} \ .$$

Therefore,

$$e^{tA} = V \begin{bmatrix} e^{t\mu_1} & 0 \\ 0 & e^{t\mu_2} \end{bmatrix} V^{-1} \ .$$

Since both eigenvalues are negative, $\lim_{t\to\infty} e^{t\mu_1} = 0$ and $\lim_{t\to\infty} e^{t\mu_2} = 0$. Therefore,

$$\lim_{t\to\infty} e^{tA} = 0 \ .$$

• *It follows that because both eigenvalues of A are negative and distinct,*

$$\lim_{t\to\infty} e^{tA}\mathbf{z}_0 = 0 \ ,$$

no matter what the starting point $\mathbf{z}_0$ is. Hence $\mathbf{x}_0 + e^{tA}\mathbf{z}_0$ remains in a small neighborhood of $\mathbf{x}_0$ for all time, and the approximation (3.15); i.e.,

$$\mathbf{x}(t) = \mathbf{x}_0 + \epsilon^{tA}\mathbf{x}_0$$

remains valid for all t, so that we expect

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0 \ . \qquad (3.17)$$

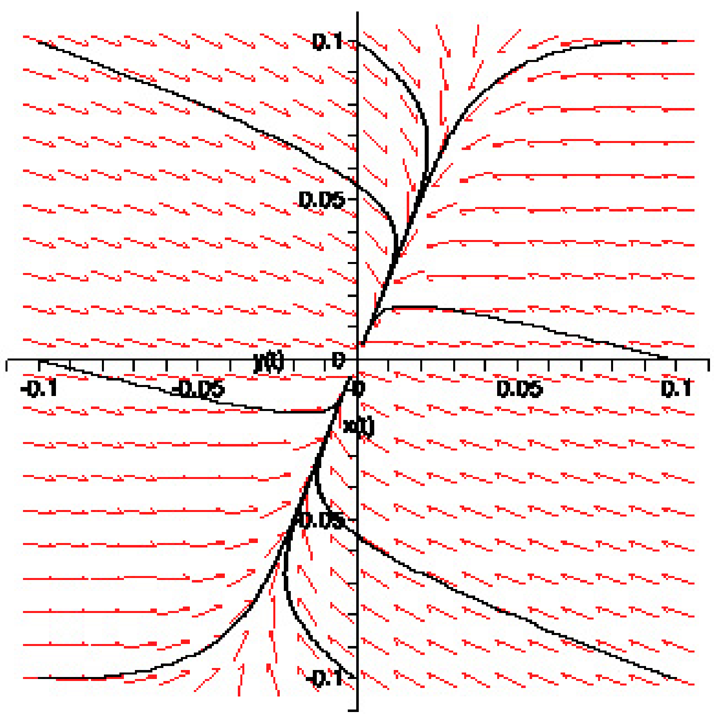
In other words, if we start out at a point close to $\mathbf{x}_0$, say $\mathbf{x}_0 + \mathbf{z}_0$ with $\mathbf{z}_0$ small, then we would expect that the solutions $\mathbf{x}(t)$ would satisfy $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$. *That is, all flow liness that start near $\mathbf{x}_0$, end up at $\mathbf{x}_0$.* This is exactly what we saw in our graphs of flow lines in the previous section. In fact, you see that $\mathbf{z}_0$ does not have to be all that small in this example.

We remark that we have not given a formal proof of anything yet; our conclusions are based on the approximation (3.9). However, as $|\mathbf{z}(t)|$ gets smaller smaller this approximation becomes more and more accurate. For this reason, (3.17) will hold exactly.

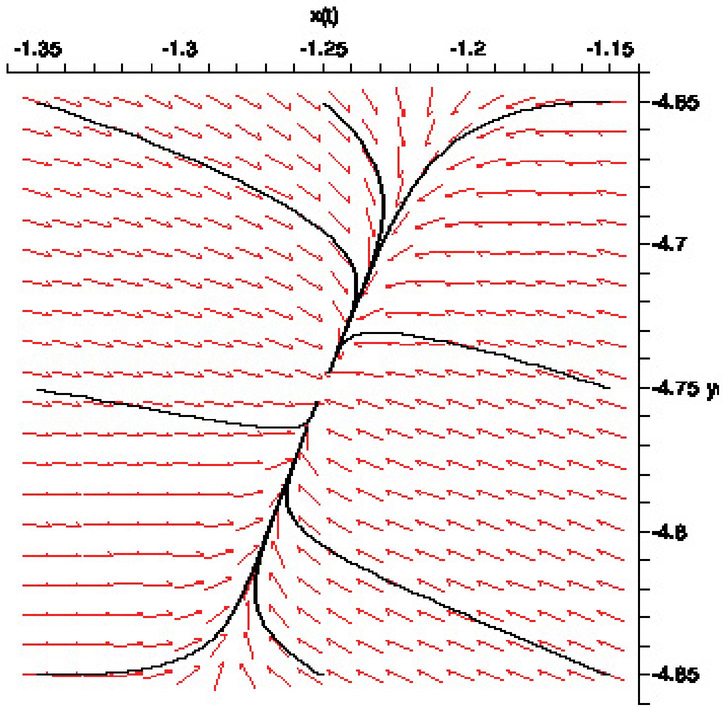
Let's draw a phase portrait showing the solutions of (3.12) for the initial data points $\mathbf{z}_0$ given by

$$\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \quad \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} \quad \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix} . \tag{3.18}$$

These are the eight "compass points" around the origin. The graph showing the flow lines starting from them is:



Now let's plot the phase portrait of the true equation $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$ starting from the eight points $\mathbf{x}_0 + \mathbf{z}_0$ where $\mathbf{z}_0$ is given, in succession, by the eight points in (3.18). The result is



The only difference is that the pattern of the lines in the second graph is "shifted" from being centered on the origin to being centered on the equilibrium point $-\frac{1}{4}\begin{bmatrix} 5 \\ 19 \end{bmatrix}$. The message is this:

• *If you want to know what the pattern of flow lines looks like near an equlibrium point, compute the linearization of the system about the equilibrium point, and graph its flow lines. What you see is just a shifted version of the pattern of flow lines near the equilibrium point.*

Now, let's apply this method to determine the pattern of flow lines near the other three equilibrium points.

First consider $\mathbf{x}_0 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. Computing $J_{\mathbf{F}}(\mathbf{x}_0)$ we find

$$J_{\mathbf{F}}(\mathbf{x}_0) = \frac{1}{\sqrt{17}}\begin{bmatrix} -13 & -13 \\ -30 & -15 \end{bmatrix} .$$

The eigenvalues are

$$\mu_{\pm} = -\frac{14}{\sqrt{17}} \pm \sqrt{23} .$$

These are about 1.40 and -8.19. In particular, one is positive and one is negative. Let $\mathbf{v}_+$ and $\mathbf{v}_-$ be eigenvectors for μ_+ and μ_-, and let $V = [\mathbf{v}_+, \mathbf{v}_-]$. Then

$$A = V\begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix} V^{-1} ,$$

and so

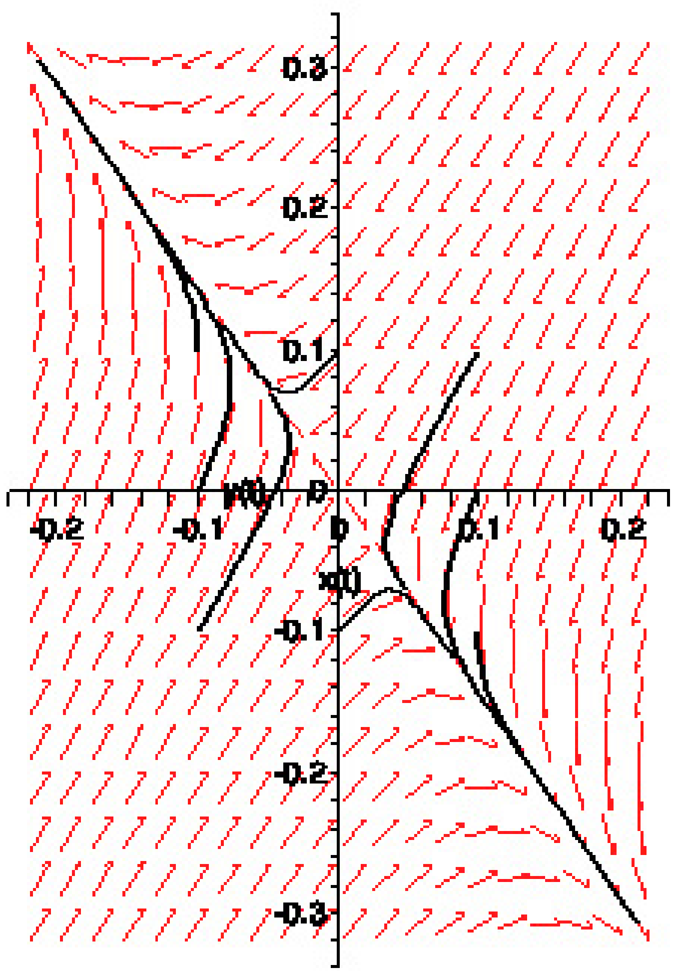
$$e^{tA} = V \begin{bmatrix} e^{t\mu_+} & 0 \\ 0 & e^{t\mu_-} \end{bmatrix} V^{-1} .$$

Since μ_+ is positive, $e^{t\mu_+}$ grows rapidly as t increases, and since μ_- is negative, $e^{t\mu_-}$ shrinks rapidly as t increases. Hence for large values of t, $e^{t\mu_-}$ is negligible with respect to $e^{t\mu_+}$. Hence we have

$$e^{tA} \approx V \begin{bmatrix} e^{t\mu_+} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} .$$

This matrix definitely does not satisfy $\lim_{t\to\infty} e^{tA} = 0$. In fact, it "blows up" as t tends to infinity.

Let's graph the flow lines starting from the eight points $\mathbf{x}_0 + \mathbf{z}_0$ where $\mathbf{z}_0$ is given, in succession, by the eight points in (3.18). Here is the graph:



You see that all of the points get "swept away" from this equilibrium point. (To se the direction of motion along the flow lines, look at the directions of the nearby arrows in the phase portrait). In fact, they exit away from it along the line in the direction of $\mathbf{v}_+$ which is approximately $\begin{bmatrix} -0.692 \\ 1 \end{bmatrix}$. The line spanned by this vector is the "exit line" from this equilibrium point: all eight starting points quickly head to this line, and then move away along it, away from the equilibrium point.

The other two equilibrium points are like this one: At each of them, there is one positive and one negative eigenvalue. Typical points nearby these equilibrium points are "swept away" from them along the line spanned by the eigenvector with the positive eigenvalue.

Such points are *unstable*: If you start at a typical point near them, you are "swept away". In contrast, the first point we examined is *stable*: If you start at a typical point near them, you are "pulled in".

The difference between stability and instability of an equilibrium point lies in the eigenvalues of the Jacobian at the equilibrium point: If both eigenvalues are strictly negative, then the equilibrium point is stable. If you start at a typical point near them, you are "pulled in".

On the other hand, if either is strictly positive, then the then the equilibrium point is unstable. If you start at a typical point near them, you are "swept away". We will refine this classification of equilibrium points in the next subsection.

3.3: Stability and eigenvalues

Definition (Stability) An equilibrium point $\mathbf{x}_0$ of a vector field $\mathbf{F}$ is *stable* in case there is an $r > 0$ so that for all $\mathbf{z}_0$ with $|\mathbf{z}_0| \le r$, the solution of

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 + \mathbf{z}_0$$

satisfies

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0 \ .$$

The definition is such that if you start the motion close enough to an equilibrium point, the motion eventually gets "pulled in" to the equilibrium point.

Example 3 (Stable equilibrium) The equilibrium point $-\frac{1}{4}\begin{bmatrix} 5 \\ 19 \end{bmatrix}$ is stable for the vector field $\mathbf{F}$ given by (3.1). So far, we have seen this in pictures. We will soon see how to demonstrate the stability by explicit computation.

We will now define instability to express the intuitive idea that if you start nearby an unstable equilibrium point, you *can* get "swept away" no matter how close you start. To be precise, we fix some number $R > 0$, and we will consider $\mathbf{x}(t)$ to have been swept away from $\mathbf{x}_0$ if $|\mathbf{x}(t) - \mathbf{x}_0| \ge R$. If you can find initial data $\mathbf{x}_0 + \mathbf{z}_0$ that gets swept away for arbitrarily small $\mathbf{z}_0$, then you have what we will call instability.

Definition (Instability) An equilibrium point $\mathbf{x}_0$ of a vector field $\mathbf{F}$ is *unstable* in case there is an $R > 0$ so that for all $r > 0$ there is a $\mathbf{z}_0$ with $|\mathbf{z}_0| \le r$ such the solution of

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 + \mathbf{z}_0$$

satisfies $|\mathbf{x}(t) - \mathbf{x}_0| \geq R$ for some finite t.

Example 4 (Unstable equilibrium) The equilibrium point $\mathbf{x}_0 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ is unstable for the vector field $\mathbf{F}$ given by (3.1). So far, we have seen this in pictures. We will soon see how to demonstrate the stability by explicit computation.

The way the definitions are set up, there are equilibrium points that are neither stable nor unstable.

Example 5 (Neither stable nor unstable) Consider

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 0 & y \\ -x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} \ .$$

Evidently, $\mathbf{x} = 0$ is an equilibrium point. Since this is a linear system; i.e., since $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the solution of

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{z}_0$$

is given by $e^{tA}\mathbf{z}_0$. Now, we have computed e^{tA} in the first section, and we found it to be a 2×2 rotation matrix for each t. In particular, it does not change the length of $\mathbf{z} - 0$, and so

$$|\mathbf{x}(t)| = |\mathbf{z}_0|$$

for all $t > 0$. We do not have stability since no matter how small r is chosen, we can choose $\mathbf{z}_0 = r\mathbf{e}_1$ so that $|\mathbf{z}_0| = r$. We then have $|\mathbf{x}(t)| = r$ for all t, and not $\lim_{t\to\infty} |\mathbf{x}(t)| = 0$. So this is not stable.

On the other hand, no matter how small we pick $R > 0$, if we start at $\mathbf{z}_0$ with $|\mathbf{z}_0| < R/2$, we have for the corresponding solution that $|\mathbf{x}(t)| < R/2$ for all t, so initial data this close never gets "swept away". Hence this equilibrium point is not unstable either. Such equilibria are sometimes called *marginally stable*

If the system is linear, then it is easy to determine stability or instability by looking at eigenvalues. Indeed, consider any linear vector field $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$. Then $\mathbf{x}_0 = 0$ is an equilibrium point. Let μ_1 and μ_2 be the eigenvalues of A. Suppose that A is diagonalizable, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors of A. Let $V = [\mathbf{v}_1, \mathbf{v}_2]$. Then

$$A = V \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} V^{-1} \ ,$$

and so

$$e^{tA} = V \begin{bmatrix} e^{t\mu_1} & 0 \\ 0 & e^{t\mu_2} \end{bmatrix} V^{-1} \ .$$

It might be that μ_1 and μ_2 are complex, in which case they are a complex conjugate pair. That is, there are real numbers a and b so that $\mu_1 = a + ib$ and $\mu_2 = a - ib$. Since

$$e^{t(a \pm ib)} = e^{ta}e^{\pm itb}$$

and since $|e^{\pm itb}| = 1$, we have that $\lim_{t\to\infty} e^{t(a\pm ib)} = 0$ if a is strictly negative, and that $\lim_{t\to\infty} e^{t(a\pm ib)}$ "blows up" if a is strictly positive. Similarly, if μ_1 is real, then

$\lim_{t\to\infty} e^{t\mu_1} = 0$ if μ_1 is strictly negative, and that $\lim_{t\to\infty} e^{t\mu_1}$ "blows up" if μ_1 is strictly positive. The same conclusions hold for μ_2. Therefore, if the real part of both eigenvalues are both strictly negative, then

$$\lim_{t\to\infty} e^{tA} = V\left(\lim_{t\to\infty}\begin{bmatrix} e^{t\mu_1} & 0 \\ 0 & e^{t\mu_2}\end{bmatrix}\right)V^{-1} = 0\ .$$

This certainly implies that for *all* initial data $\mathbf{z}_0$, the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{z}_0$ satisfies $\lim_{t\to\infty}\mathbf{x}(t) = 0$. Hence in this case, $\mathbf{x}_0 = 0$ is a stable equilibrium point.

On the other hand, if either eigenvalue has a strictly positive real part then e^{tA} "blows up" as t increases. In this case, we can find arbitrarily small initial data $\mathbf{z}_0$ for which the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{z}_0$ becomes arbitrarily large. Hence in this case, $\mathbf{x}_0 = 0$ is an unstable equilibrium point.

Our analysis was carried out under the assumption that A is diagonalizable. If that is not the case, one has to use a similar argument based on generalized eigenvectors. The conclusion is the same, and the result is the following:

Theorem 1 (Stability of linear systems) *The equilibrium point $\mathbf{x}_0 = 0$ of a linear vector field $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ is stable in case each of the eigenvalues of A has a strictly negative real part, and is unstable in case at least one has a strictly positive real part*

Now the good news is that stability and instability are "robust" properties as we have defined them. If you change the vector field a little bit, you do not change whether the motion pulls things in, or sweeps them away. In fact, this is why we left out borderline cases like the one in Example 5. This means that the approximation that we make in linearizing a system does not affect the stability or instability, and we can check for these properties by applying Theorem 1 to the linearization. The hypothesis about second derivatives in the following theorem is there so that we can apply Taylor's theorem with remainder to show that the errors in the linearization are not enough to affect stability. We will not prove this in detail.

Theorem 2 (Stability of non linear systems) *Let $\mathbf{x}_0$ be an equilibrium point for a vector field $\mathbf{F}$. Suppose the entries of $\mathbf{F}$ are twice continuously differentiable. Then if each of the eigenvalues of $J_{\mathbf{F}}(\mathbf{x}_0)$ has a strictly negative real part, $\mathbf{x}_0$ is a stable equilibrium point. If any of the eigenvalues of $J_{\mathbf{F}}(\mathbf{x}_0)$ has a strictly positive real part, then $\mathbf{x}_0$ is unstable*

Problems

1. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} (x-y)(1-x^2) \\ -y^3 - x \end{bmatrix}\ .$$

Find all equilibrium points, and determine whether they are stable or unstable.

3. Consider the vector field $\mathbf{F}(\mathbf{x})$ where $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} (x^2-y)(1-x) \\ -3y + x \end{bmatrix}$. Find all equilibrium points, and determine whether they are stable or unstable.

3. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} (x-y)(2y-x) \\ (3y-x)(x-1) \end{bmatrix} .$$

Find all equilibrium points, and determine whether they are stable or unstable.

4. Consider the vector field $\mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x^2 - y^2 \\ 2y - 3x - b \end{bmatrix} .$$

Find all equilibrium points, and determine whether they are stable or unstable.

Section 4: Gradient flow

4.1: Gradient vector fields

If f is any real valued function on $\mathbb{R}^n$, we can define a vector field $\mathbf{F}$ on $\mathbb{R}^n$ by

$$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x}) \ .$$

Definition (Gradient vector field) A vector field $\mathbf{F}$ is a *gradient vector field* if and only if there is a real valued function f so that for all $\mathbf{x}$, $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$. such function f is called a *potential* for $\mathbf{F}$.

Since adding a constant to a function does not affect its gradient, if f is a potential, then so is $f + c$ for any constant c.

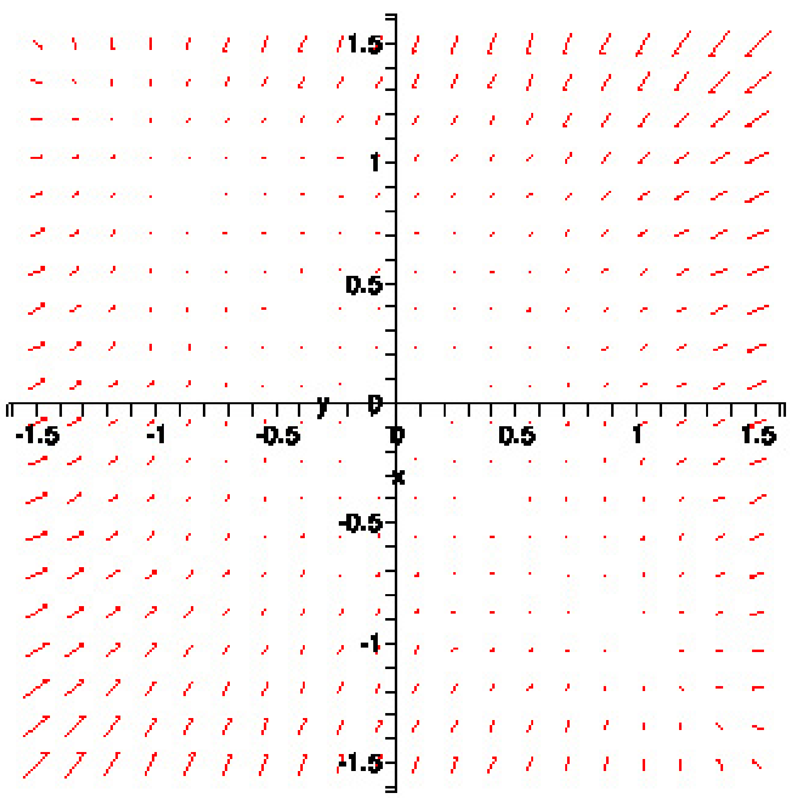
Example 1 (Gradient vector field) Consider the function

$$f(x, y) = -x^4 - y^4 - 4xy \ .$$

Then

$$\nabla f(\mathbf{x}) = -4 \begin{bmatrix} x^3 + y \\ y^3 + x \end{bmatrix} \ .$$

Here is a plot of the of the vector field $= \nabla \mathbf{f}$:



If $\mathbf{F}$ is a gradient vector field with potential f, then $\mathbf{x}_0$ is an equilibrium point of $\mathbf{F}$ if and only if $\mathbf{x}_0$ is a critical point of f. Indeed, by the definitions. $\mathbf{F}(\mathbf{x}_0) = 0$ if and only if $\nabla f(\mathbf{x}) = 0$.

Moreover,

$$J_{\mathbf{F}} = J_{\nabla f} = \text{Hess}_f \ . \tag{4.1}$$

Since the Hessian is symmetric, the eigenvalues will be real. If $\mathbf{x}_0$ is an equilibrium point of $\mathbf{F}$, and if both eigenvalues of $J_{\mathbf{F}} = \text{Hess}_f$ are positive, then $\mathbf{x}_0$ is a local minimum of the potential f, and is an unstable equilibrium point of $\mathbf{F}$. If both are negative, then $\mathbf{x}_0$ is a local maximum of the potential f, and is a stable equilibrium point of $\mathbf{F}$. If one is positive and the other is negative, then then $\mathbf{x}_0$ is a saddle point of the potential f, and is an unstable equilibrium point of $\mathbf{F}$.

Example 2 (Equilibrium points for gradient flow and critical points) Consider the function $f(x,y) = -x^4 - y^4 - 4xy$ as in the previous example. Then $\nabla f(\mathbf{x}) = -4\begin{bmatrix} x^3 + y \\ y^3 + x \end{bmatrix}$. At a critical point, we must have

$$x^3 y = 0 \qquad \text{and} \qquad y^3 + x = 0 \ .$$

You can easily check that $\mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are critical points of f. With a bit more thought, you can see that they are the only critical points. At both of these critical points we have

$$J_{\nabla f} = \text{Hess}_f = -4\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \ .$$

The eigenvalues are -16 and -8. These are both strictly negative. Hence both $\mathbf{x}_0$ and $\mathbf{x}_1$ are local maxima of f, and stable equilibrium points for the vector field $\mathbf{F} = \nabla f$.

If you look back at the field plot that we produced above, you will see the arrows pointing in toward $\mathbf{x}_0$ and $\mathbf{x}_1$ near each of them. This is the pictoral signature of stability.

We can understand this connection between minima and maxima on the one hand, and stable and unstable equilibrium points better if we think in terms of the flow lines generated by solving

$$\mathbf{x}'(t) = \nabla f(\mathbf{x}(t)) \ . \tag{4.2}$$

That is, we take $\mathbf{F} = \nabla f$. This is the equation for *steepest ascent gradient flow* generated by the potential f.

The direction of flow is is the direction of the gradient, which is to say: straight uphill. Indeed, if $\mathbf{x}(t)$ satisfies (4.2), we have from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) &= x'(t) \cdot \nabla f(\mathbf{x}(t)) \\ &= \nabla f(\mathbf{x}(t)) \cdot \nabla f(\mathbf{x}(t)) \\ &= |\nabla f(\mathbf{x}(t))|^2 \ .\end{aligned}$$

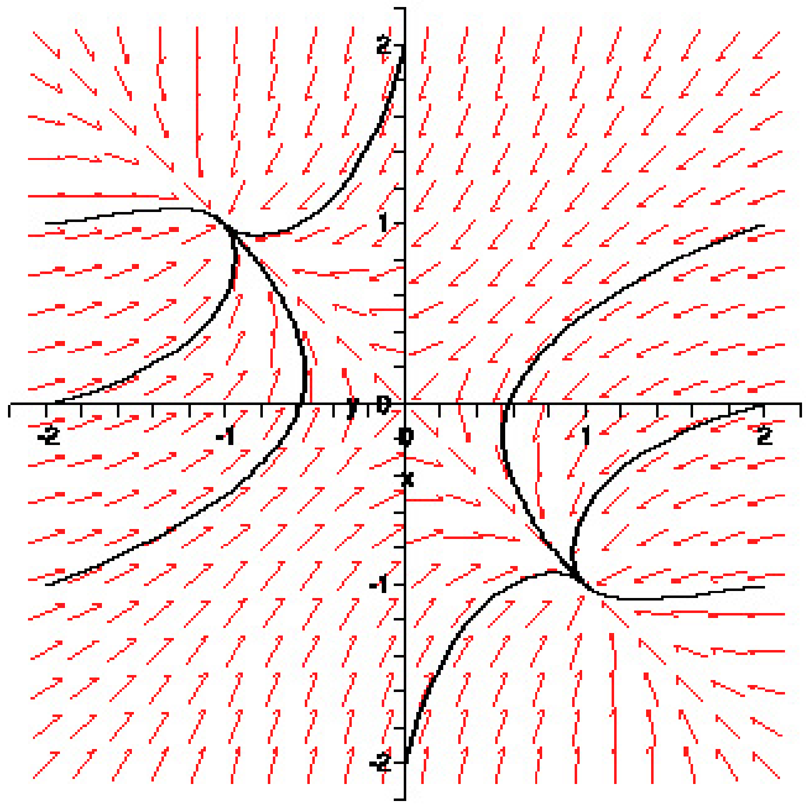
As you move along the flow lines, the value of f steadily increases. If you arrive at a local maximum, then you must stop there, since if you move, f must increase, but you are already at a local maximum. If you start near, but not exactly at, a local maximum, the flow will carry you straight uphill to the local maximum. This is the reason that strict local maximum critical points of f are stable equilibrium points of $\mathbf{F} = \nabla f$.

Example 3 (Flow lines for steepest ascent) Again, consider the function $f(x,y) = -x^4 - y^4 - 4xy$.

Here are the flow lines generated by starting at the eight initial points

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and running the flow for $0 \le t \le 2$



Each of the flow lines has converged on one or the other of the equilibrium points $\mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. It may look like there is a smooth curve running from $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ that passes through $\mathbf{x}_0$, but this is not the case: You can tell that the line flow in and not through by looking at the directiosn of the nearby arrows. They all point inwards. Hence what you really have is two flow lines approaching $\mathbf{x}_0$ from opposite sides. As they approach $\mathbf{x}_0$, they progress along the flow line slows down to a crawl. They crawl on up toward the top of the hill to stop there – it is an equilibrium point. They *do not* proceed down the other side of the hill.

On the other hand if you start near, but not exactly at, a local minimum, the flow will carry you straight uphill away from the local mnimum. This is the reason that strict local minimum critical points of f are unstable equilibrium points of $\mathbf{F} = \nabla f$.

On the other hand, consider the equation

$$\mathbf{x}'(t) = -\nabla f(\mathbf{x}(t)) \ . \tag{4.3}$$

That is, we take $\mathbf{F} = -\nabla f$. This is the equation for *steepest descent gradient flow* generated by the potential f.

The direction of flow is is *opposite* the direction of the gradient, which is to say: straight downhill. Indeed, if $\mathbf{x}(t)$ satisfies (4.2), we have from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{x}(t)) &= x'(t)\cdot\nabla f(\mathbf{x}(t))\\ &= -\nabla f(\mathbf{x}(t))\cdot\nabla f(\mathbf{x}(t))\\ &= -|\nabla f(\mathbf{x}(t))|^2\ .\end{aligned}$$

As you move along the flow lines, the value of f steadily decreases. If you arrive at a local minimum, then you must stop there, since if you move, f must decrease, but you are already at a local minimum. If you start near, but not exactly at, a local minimum, the flow will carry you straight downhill to the local minimum. Thus, strict local minimum critical points of f are stable equilibrium points of $= -\nabla \mathbf{f}$.

These observations can be used to local maxima and minima: Suppose that we are trying to find local minima of $f(\mathbf{x})$. We can pick a starting guess $\mathbf{x}_0$, and then solve

$$\mathbf{x}'(t) = -\nabla f(\mathbf{x}(t)) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0\ . \tag{4.4}$$

When the flow line reaches an equilibrium point and stops, assuming it does not just "flow away" forever, the stopping point will be an equilibrium point. In the same way, one can look for local maxima using steepest ascent gradient flow.

By itself, this would not be a very good method, since under either type of gradient flow $|\mathbf{x}'(t)| = |\nabla f(\mathbf{x}(t))|$ is small whenver $\mathbf{x}(t)$ is close to a critical point. That is, the flow carries you toward the critical point, but it slows down as it gets close, and the closer it gets, the more it slows down. Once the slowing down has occured though, one can switch to Newton's method. Recall that Newton's method requires a good starting guess. What we can use gradient flow for then is to find the starting guess.

• *While gradient flow "slows down" in the neighborhood of a critical point, it can be used to find good starting guesses for Newton's method in the search for local minima or maxima. If you are looking for local minima, run steepest descent gradient flow until it slows down to a crawl, and then use the final point as a starting point for Newton's method. If you are looking for local maxima, run steepest asscent gradient flow until it slows down to a crawl, and then use the final point as a starting point for Newton's method.*

4.2: Perpendicular gradient flow and level curves

Let f be a function on $I\!R^2$, and consider the *perpendicular gradient field*

$$\mathbf{F}(\mathbf{x}) = (\nabla f(\mathbf{x}))^\perp\ . \tag{4.5}$$

If $\mathbf{x}(t)$ satisfies

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0\ , \tag{4.6}$$

we have from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{x}(t)) &= x'(t)\cdot\nabla f(\mathbf{x}(t))\\ &= (\nabla f(\mathbf{x}(t)))^{\perp}\cdot\nabla f(\mathbf{x}(t))\\ &= 0\ .\end{aligned}$$

That is, f is constant along the flow line through $\mathbf{x}_0$ of this vector field. This means that the flow line is (part of) a level curve of f.

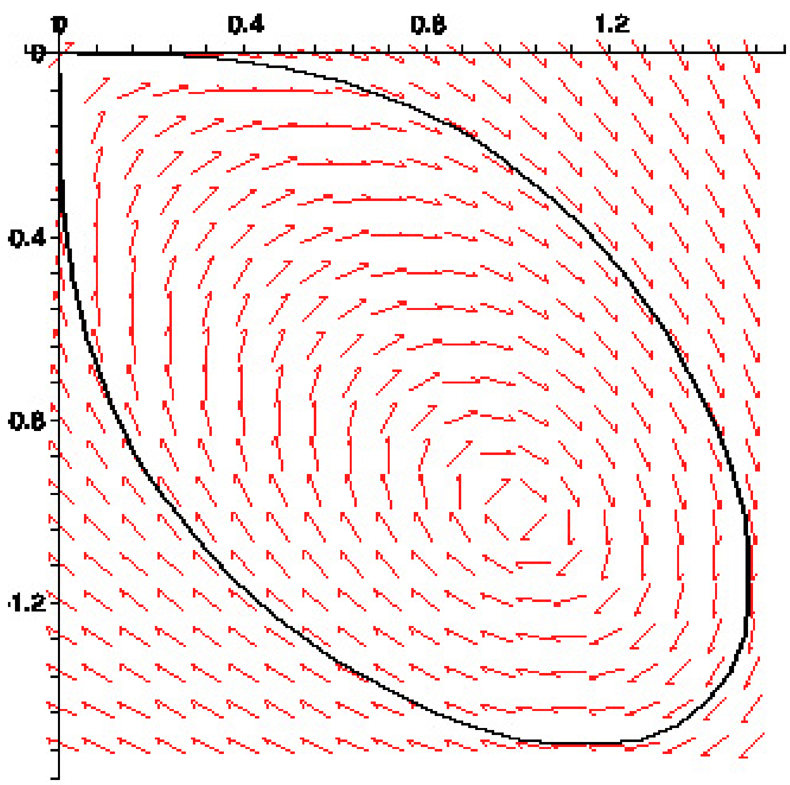
You may or may not get the whole level curve this way. For instance, if there is a critical point of f along the level curve, that at this critical point $\nabla f = 0$, and hence so does $(\nabla f)^{\perp}$. Therefore, the flow stops at this critical point; it cannot pass beyond it. If you follow the flow backwards, you continue tracing out the level curve. If there is a critical point in this direction, you will grind to a halt as you approach it. What you will have traced out is an arc of the level curve with critical points at the ends.

Example 4 (Level curves and perpendicular gradient flow) Again, consider the function $f(x,y) = -x^4 - y^4 - 4xy$. You can easily check that

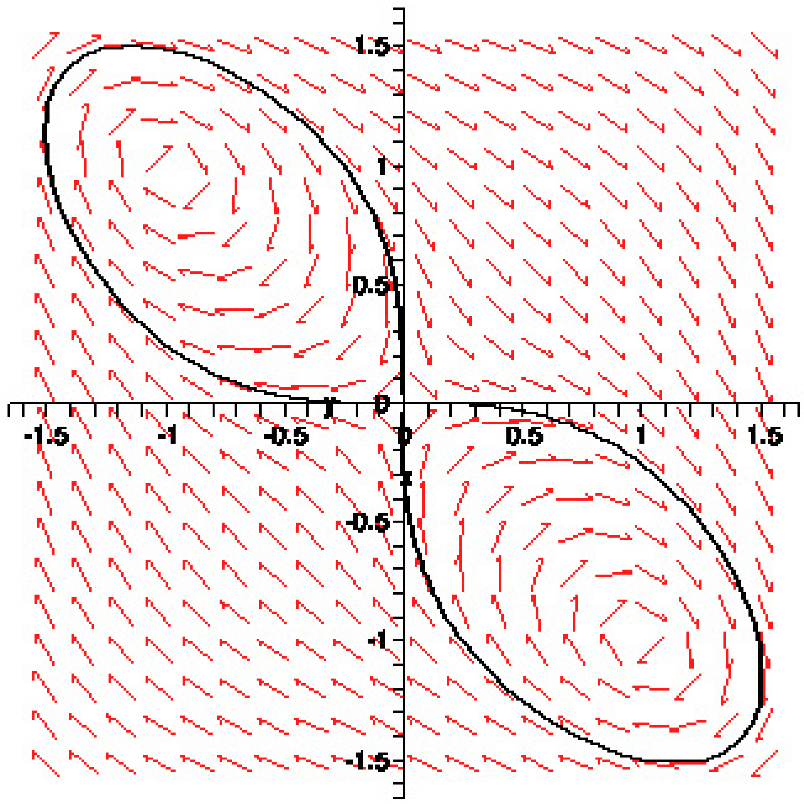
$$f(-\sqrt{2},\sqrt{2}) = f(\sqrt{2},-\sqrt{2}) = 0$$

Hence the two points $\sqrt{2}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\sqrt{2}\begin{bmatrix}-1\\1\end{bmatrix}$ belong to the level curve given implicitly by $f(x,y) = 0$.

If we start from the first of these, and draw the flow line for perpendicular gradient flow through this point for $-1 \le t \le 1$, we find:



The ends of the path, continued for all t, would be in the critical point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We never get around to the point $\sqrt{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ on this arc of the path. But using this as a satrting point, we pick up the other arc of the level set:



what we have seen in this example is very useful:

• *By solving the perpendicular gradient flow equation (4.6), you find an arc of the level curve of f through $\mathbf{x}_0$. The endpoints of the arc, if any, will be critical points of f.*

This observation allows us to use Euler's method to find level curves. It also has other uses. for example, we can use it to deduce a formula for the curvature of an implicitly defined curve.

Let f be a given function on $I\!R^2$, and let $\mathbf{x}_0$ be a given point in $I\!R^2$. Let $\mathbf{x}(t)$ be the solution of (4.6), So that this is the level curve of f through $\mathbf{x}_0$. We know that the curvature of this curve at $\mathbf{x}_0$ is given by

$$\kappa = \frac{|\mathbf{x}''(0) \cdot (\mathbf{x}'(0))^{\perp}|}{|\mathbf{x}'(0)|^3} \ . \tag{4.7}$$

We compute, using the chain rule,

$$\mathbf{x}'' = (\mathbf{x}')' = \left((\nabla f(\mathbf{x}))^{\perp}\right)' = J_{(\nabla f)^{\perp}}\mathbf{x}'(t)$$

and of course

$$(\mathbf{x}')^{\perp} = (\nabla f(\mathbf{x}))^{\perp\perp} = -\nabla f(\mathbf{x}) \ .$$

To carry our the computation in an intelligible manner, it helps to have a simpler notation for partial derivatives. A common practice is to use subscripts to denote partial derivatives. We define f_x to be $\partial f/\partial x$ and f_y to be $\partial f/\partial x$:

$$f_x = \frac{\partial f}{\partial x} \qquad \text{and} \qquad f_y = \frac{\partial f}{\partial y} \ .$$

Likewise,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \qquad \text{and} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} \ .$$

In this notation, $(\nabla f)^\perp = \begin{bmatrix} -f_y \\ f_x \end{bmatrix}$ and so

$$J_{(\nabla f)^\perp} = \begin{bmatrix} -f_{yx} & -f_{yy} \\ f_{xx} & f_{xy} \end{bmatrix} \ .$$

Therefore,

$$\mathbf{x}'' = \left[J_{(\nabla f)^\perp} \right] (\nabla f(\mathbf{x}))^\perp = \begin{bmatrix} -f_{yx} & -f_{yy} \\ f_{xx} & f_{xy} \end{bmatrix} \begin{bmatrix} -f_y \\ f_x \end{bmatrix} = \begin{bmatrix} f_{yx} f_y - f_{yy} f_x \\ -f_{xx} f_y + f_{xy} f_x \end{bmatrix} \ .$$

Taking the dot product with $(\mathbf{x}')^\perp = -\nabla f$, we have

$$\begin{aligned} \mathbf{x}'' \cdot (\mathbf{x}')^\perp &= \begin{bmatrix} f_{yx} f_y - f_{yy} f_x \\ -f_{xx} f_y + f_{xy} f_x \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ &= 2 f_{xy} f_x f_y - f_{xx} (f_y)^2 - f_{yy} (f_x)^2 \ . \end{aligned}$$

Finally, we have that the curvature κ is obtained by dividing this by $|\nabla f|^3$. Of course, this is not possible at a critical points, but then in general, it does not make sense to talk about "the level curve of f" through a critical point. Away from critical points, the implicit function theorem guarantees that it does make sense.

This gives us the following result:

Theorem 1 *Let f be a twice continuously differentiable function. Then at any point $\mathbf{x}_0$ that is not a critical point, the curvature κ of the level curve of f through $\mathbf{x}_0$ is given by*

$$\kappa = \frac{\left| 2 f_{xy} f_x f_y - f_{xx} (f_y)^2 - f_{yy} (f_x)^2 \right|}{((f_x)^2 + (f_y)^2)^{3/2}} \ , \tag{4.8}$$

evaluated at $\mathbf{x}_0$

Example 5 Let $f(x, y) = x^2 + y^2$. Then $f_x = 2x$, $f_y = 2y$, $f_{xx} = f_{yy} = 2$, and $f_{xy} = 0$. Hence

$$2 f_{xy} f_x f_y - f_{xx} (f_y)^2 - f_{yy} (f_x)^2 = -2(2x)^2 - 2(2y)^2 = -8|\mathbf{x}|^2$$

while

$$((f_x)^2 + (f_y)^2)^{3/2} = 8|\mathbf{x}|^3 \ .$$

Hence the formula gives

$$\kappa = 1/|\mathbf{x}| \ .$$

Notice that that level curve of f through $\mathbf{x}$ is the circle of radius $r = |\mathbf{x}|$, and we have computed earlier, using an explicit parameterization, that the curvature of a circle of radius r is $1/r$. So the result we have obtained from (4.8) is what it should be.

Theorem 1 is particularly useful when an explicit parameterzation is cumbersome or impossible to find. To use (4.7), you must have a parameterization of the level curve. Formula (4.8) sidesteps this difficulty.

Example 6 Let $f = -x^4 - y^4 - 4xy$ as in our earlier examples. Computing

$$\frac{2f_{xy}f_xf_y - f_{xx}(f_y)^2 - f_{yy}(f_x)^2}{((f_x)^2 + (f_y)^2)^{3/2}} \ ,$$

we find

$$\frac{10\,x^3y^3 + x^4 + y^4 - 2\,xy + 3\,x^2y^6 + 3\,y^2x^6}{(x^6 + 2\,x^3y + y^2 + y^6 + 2\,y^3x + x^2)^{3/2}} \tag{4.9}$$

As we have seen above, the points $\sqrt{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\sqrt{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ belong to the level curve given implicitly by $f(x, y) = 0$, which we have already plotted above.

Evaluating the ratio (4.9) at either of them, we find that the curvature is $7/2$. Hence the radius of curvature at these points is $2/7$, which, if you inspect the graph, looks pretty reasonable.

Notice that the curvature function (4.9) is not continuous at the origin. For x and y both very close to zero, we can neglect everything except the terms involving the least total power in x and y:

$$\frac{10\,x^3y^3 + x^4 + y^4 - 2\,xy + 3\,x^2y^6 + 3\,y^2x^6}{(x^6 + 2\,x^3y + y^2 + y^6 + 2\,y^3x + x^2)^{3/2}} \approx \frac{-2xy}{(x^2 + y^2)^{3/2}} \ .$$

If one approaches the origin tangential either the line $x = 0$ or $y = 0$, the limit of the ratio is zero since the numerator vanishes identically. This is what our level curves do, and if you look at the graph, the level curve does indeed seem to have zero curvature near the origin.

However, nearby, there are points through which the level curve has very high curvature. Consider for example $\begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}$. Evaluating (4.9) at this point we find the curvature to be about 70.682.... Look at the arrows, and try to visualize this level curve so that you see its large curvature.

Problems

Problem 1 Let $f(x, y) = x^3 + y^3 - 4xy$. Let $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$.

(a) Find all of the equilibrium points of f and determine whether they are stable or unstable.

(b) Use Euler's scheme and perpendicular gradient flow to graph the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(c) Compute the curvature of the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ at $\mathbf{x}_0$.

Problem 2 Let $f(x, y) = x^4 + y^4 - 2x^2y^2$. Let $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$.

(a) Find all of the equilibrium points of f and determine whether they are stable or unstable.

(b) Use Euler's scheme and perpendicular gradient flow to graph the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Compute the curvature of the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at $\mathbf{x}_0$.

Problem 3 Let $f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2)$. Let $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$.
(a) Find all of the equilibrium points of f and determine whether they are stable or unstable.
(b) Use Euler's scheme and perpendicular gradient flow to graph the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$.

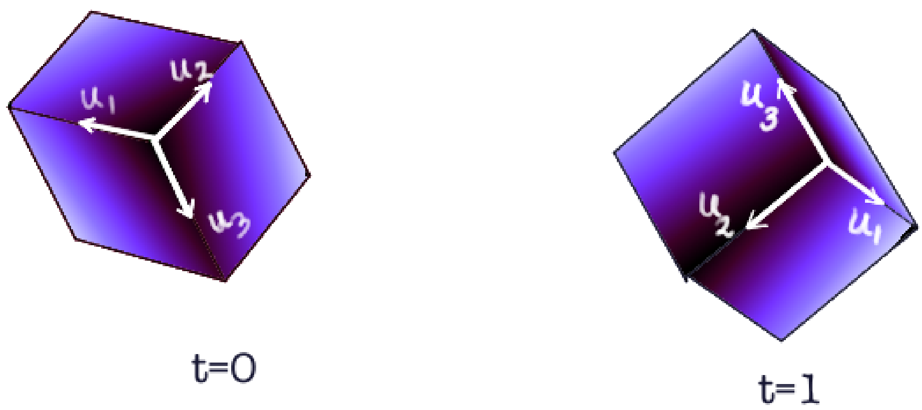
(c) Compute the curvature of the level curve of $f(x, y)$ passing through $\mathbf{x}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ at $\mathbf{x}_0$.

Problem 4 Explain why a critical point of perpendicular gradient flow is never stable.

Section 5: Rotations and Rigid Body Motion

5.1: Rotations in $I\!R^3$ and the orientation of rigid body

Imagine a solid, rigid object moving in three dimensional space. To keep the picture simple, suppose the object is "box shaped". Here is a picture showing the box shaped object at two times: $t = 0$ and $t = 1$:



As it moves, the box carries with it a "reference frame" of three unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Their common origin is some "reference point" $\mathbf{b}$ in the rigid body. Here we have chosen a particular corner. One might well choose the center of mass, or any other particular point.

The vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ provide a basis that can be used to locate any point in the rigid body relative to $\mathbf{b}$. Suppose that a point $\mathbf{p}$ in the body is given by

$$\mathbf{p} = \mathbf{b} + \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3 \ .$$

The vector $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ describes the vector $\mathbf{p}$ relative to the reference point $\mathbf{b}$, and is called the *body coordinate vector of* $\mathbf{p}$.

If you know the body coordinates of key points in the body – say the vertices of the cube – and you know the reference frame – $\mathbf{b}$, together with $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, you know eveything about how the body is situated in $I\!R^3$.

The important fact about this description is that as the object moves, $\mathbf{b}$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ change, the body coordinates α, β and γ do not. This is what is means for a body to be "rigid": As the body moves, the points in it move, but the distances between them do not change – there is no stretching or compression. Therefore, if at time $t = 0$ we know the body coordinates (α, β, γ) of enough points $\mathbf{p}$ in the body relative to the frame $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ centered on $\mathbf{b}$, and if we know how $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathbf{b}$ evolve in time, we can reconstruct the exact configuration of the body at every time t in the future.

This is good news: No matter how many key points we need to describe a rigid body, we do not need to keep track their individual motion. We just need to keep track of the motion of $\mathbf{b}$, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

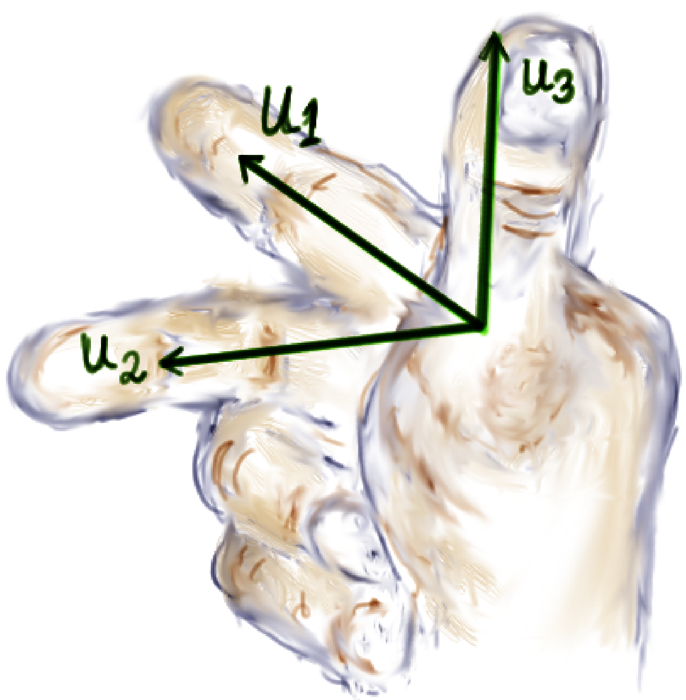
• *The essential data required to describe the motion of any rigid body in* $\mathbb{R}^3$ *is the location of a base point* $\mathbf{b}$*, together with an orthonormal basis of vectors* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$*.*

The choice of the orthonormal basis that we will embed into the rigid body at $t = 0$ is somewhat arbitrary. But it will be convenient to impose one particular requirement: We require that the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be *right handed*.

In mathematical terms, an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed if and only if

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 \ .$$

For *any* orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, $\mathbf{u}_1 \times \mathbf{u}_2 = \pm\mathbf{u}_3$ since $\mathbf{u}_1 \times \mathbf{u}_2$ is a unit vector orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. There are only two of these: $\pm\mathbf{u}_3$. If $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, the basis or right handed, and otherwise it is left handed. The following picture explains the terminology:



If you can arrange your thumb, index and middle fingers to point along the directions of an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ as in the picture, so that you index finger points in the direction of $\mathbf{u}_1$, and your middle finder points in the direction of $\mathbf{u}_2$, and your thumb points in the direction of $\mathbf{u}_3$, then that basis is right handed, or else you are *very* double jointed.

We can express the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right handed orthonormal basis in terms of the matrix $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ as follows:

First, the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal is equivalent to the fact that

$$U^t U = I \ .$$

Indeed,

$$\begin{aligned}(U^t U)_{i,j} &= (\text{row } i \text{ of } U^t) \cdot (\text{column } j \text{ of } U)\\ &= (\text{column } i \text{ of } U) \cdot (\text{column } j \text{ of } U) \ .\\ &= \mathbf{u}_i \cdot \mathbf{u}_j\end{aligned}$$

If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal, $\mathbf{u}_i \cdot \mathbf{u}_j = 1$ if $i = j$, and 0 if $i \neq j$, and *vice–versa*. This means that $U^t U = I$ if the columns of U are orthonormal and *vice–versa*.

Next, the relation between the determinant and the cross product means that

$$\begin{aligned}\det([\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]) &= \mathbf{u}_1 \times \mathbf{u}_2 \cdot \mathbf{u}_3\\ &= \mathbf{u}_3 \cdot \mathbf{u}_3\\ &= 1 \ .\end{aligned}$$

Hence, $\det(U) = 1$ if U is right handed, and $\det(U) = -1$ if U is left handed.

This motivates the study of 3×3 matrices Q with the properties that $Q^t Q = I$ and $\det(Q) = 1$. We make a definition:

Definition A 3×3 *rotation matrix* is a 3×3 matrix Q satisfying

$$Q^t Q = I \qquad \text{and} \qquad \det(Q) = 1 \ . \tag{5.1}$$

The reason for the name is this: The transformation of $I\!R^3$ induced by every 3×3 matrix Q such that $Q^t Q = I$ and $\det(Q) = 1$ is a rotation about some axis in $I\!R^3$. In the next subsection we explain this relation between right handed orthonormal bases and rotations. There are some things about rotation matrices that follow easily from the definition (5.1).

Theorem 1: *Let Q be any 3×3 rotation matrix. Then Q is invertible, and Q^{-1} is also a rotation matrix. Moreover, the product of any two rotation matrices is again a rotation matrix.*

Proof: If Q is a rotation matrix, $Q^t Q = I$, so Q^t is the left inverse of Q. But since Q is a square matrix, this means that Q^t is the inverse of Q. Now we check that Q^t satisfies (5.1): Sine taking the transpose does not change the determinant, $\det(Q^t) = \det(Q) = 1$. Also, since Q is the inverse of Q^t, $(Q^t)^t Q^t = Q Q^t = I$. Hence (5.1) is satisfied, and Q^t is also a rotation.

Next, let Q_1 and Q_2 be two rotation matrices. Then

$$\det(Q_1 Q_2) = \det(Q_1)\det(Q_2) = 1 \ ,$$

and

$$(Q_1Q_2)^t(Q_1Q_2) = Q_2(Q_1^tQ_1)Q_2 = Q_2^tQ_2 = I\ ,$$

so that Q_1Q_2 satisfies (5.1). ■

We also recall an important property of rotation matrices: For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $I\!R^3$, and any rotation matrix Q,

$$Q\mathbf{u} \times Q\mathbf{v} = Q(\mathbf{u} \times \mathbf{v})\ . \tag{5.2}$$

To see this, let $\mathbf{w}$ be any vector in $I\!R^3$, and use the relation between the cross product and the determinant:

$$\begin{aligned} Q\mathbf{u} \times Q\mathbf{v} \cdot Q\mathbf{w} &= \det([Q\mathbf{u}, Q\mathbf{v}, Q\mathbf{w}]) \\ &= \det(Q[\mathbf{u}, \mathbf{v}, \mathbf{w}]) \\ &= \det(Q)\det([\mathbf{u}, \mathbf{v}, \mathbf{w}]) \\ &= \det([\mathbf{u}, \mathbf{v}, \mathbf{w}]) \\ &= \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}\ . \end{aligned}$$

Therefore, $Q^t(Q\mathbf{u} \times Q\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{w}$ in $I\!R^3$. This means that $Q^t(Q\mathbf{u} \times Q\mathbf{v}) = \mathbf{u} \times \mathbf{v}$, so that (5.2) holds.

5.2: Rotations in $I\!R^3$ from a geometric point of view

The definition of rotation matrices that we gave in the last section was completely algebraic. What does it have to do with our geometric understanding of rotations in $I\!R^3$?

Rotations, considered as transformations of $I\!R^3$ are prime examples of "rigid transformations of $I\!R^3$":

Definition (Rigid transformation of $I\!R^3$) A transformation T from R^3 to R^3 is *rigid* in case for any $\mathbf{x}$ and $\mathbf{y}$ in R^3, we have

$$|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|\ . \tag{5.3}$$

That is, the transformation T preserves the distance between any pair of points. Such a transformation is called "rigid" since you if you connected $\mathbf{x}$ and $\mathbf{y}$ by an iron bar, the same iron bar, without any stretching, would connect $T(\mathbf{x})$ and $T(\mathbf{y})$.

Notice we are *not* assuming that T is a linear transformation. Our assumption (5.3) is geometric, and not algebraic. However:

Theorem 2 (Rigid transformations of $I\!R^3$) *Let T be any rigid transformation of $I\!R^3$. Then there is an orthogonal matrix U and a vector $\mathbf{b}$ so that for all $\mathbf{x}$ in $I\!R^3$,*

$$T(\mathbf{x}) = U\mathbf{x} + \mathbf{b}\ .$$

In fact, $\mathbf{b} = T(0)$, and and the ith column of U, $\mathbf{u}_i$, is given by $T(\mathbf{e}_i) - T(0)$.

We see from the theorem that a rigid transformation is linear if and only if $T(0) = 0$, in which case we have $T(\mathbf{x}) = U\mathbf{x}$. If $T(0) \neq 0$, then we can always define a new transformation $\tilde{T}$ by

$$\tilde{T}(\mathbf{x}) = T(\mathbf{x}) - T(0) \ .$$

Then $\tilde{T}(\mathbf{x}) - \tilde{T}(y) = T(\mathbf{x}) - T(\mathbf{y})$, so $\tilde{T}$ is also a rigid transformation, and it does have the property that $\tilde{T}(0) = 0$. If we can prove that $\tilde{T}(\mathbf{x}) = U\mathbf{x}$ for some orthogonal matrix U, and we define $\mathbf{b} = T(0)$, then we have $T(\mathbf{x}) = \tilde{T}(\mathbf{x}) + \mathbf{b} = U\mathbf{x} + \mathbf{b}$. Hence it suffices to prove the theorem under the additional assumtion that $T(0) = 0$.

Proof: Under the assumption that $T(0) = 0$, T preserves the length of vectors: For any $\mathbf{x}$ in $I\!R^3$,

$$|T(\mathbf{x})| = |T(\mathbf{x}) - T(0)| = |\mathbf{x} - 0| = |\mathbf{x}| \ .$$

Moreover, T preserves dot products: For any $\mathbf{x}$ and $\mathbf{y}$ in R^3,

$$\begin{aligned} |T(\mathbf{x}) - T(\mathbf{y})|^2 &= |T(\mathbf{x})|^2 + |T(\mathbf{y})|^2 - 2T(\mathbf{x}) \cdot T(\mathbf{y}) \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2T(\mathbf{x}) \cdot T(\mathbf{y}) \end{aligned}$$

where we have used the fact that T preserves lengths of vectors. But

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x} \cdot \mathbf{y}$$

and since $|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$, we conclude from the last two equations above that

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \ .$$

For each i with $1 \leq i \leq 3$, define $\mathbf{u}_i$ by

$$\mathbf{u}_i = T(\mathbf{e}_i) \ .$$

Then since T preserves lengths and dot products, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set of vectors in $I\!R^3$.

Therefore, $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ is an orthogonal matrix, and so U^t is the inverse of U. In particular, $UU^t = I$.

Hence for any vector $\mathbf{x}$ in R^n,

$$\begin{aligned} T(\mathbf{x}) &= (UU^t)T(\mathbf{x}) \\ &= U(U^tT(\mathbf{x})) \\ &= U \begin{bmatrix} \mathbf{u}_1 \cdot T(\mathbf{x}) \\ \mathbf{u}_2 \cdot T(\mathbf{x}) \\ \mathbf{u}_3 \cdot T(\mathbf{x}) \end{bmatrix} \\ &= (\mathbf{u}_1 \cdot T(\mathbf{x}))\mathbf{u}_1 + (\mathbf{u}_2 \cdot T(\mathbf{x}))\mathbf{u}_2 + (\mathbf{u}_3 \cdot T(\mathbf{x}))\mathbf{u}_3 \ . \end{aligned}$$

But since T preserves dot products,

$$\mathbf{u}_i \cdot T(\mathbf{x}) = T(\mathbf{e}_i) \cdot T(\mathbf{x}) = \mathbf{e}_i \cdot \mathbf{x} = x_i$$

for each i. Hence

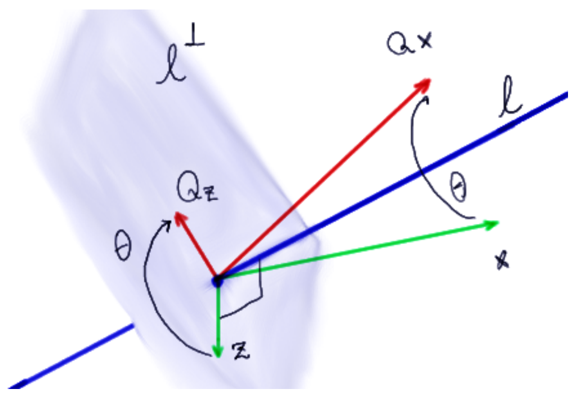
$$T(\mathbf{x}) = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 \ .$$

Then with $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, we have that $T(\mathbf{x}) = U\mathbf{x}$. ■

Now, if T is any rotation of $I\!R^3$, T certainly satisfies (5.3), and not only is $T(0) = 0$, but T leaves an entire line through the origin in place. This line ℓ is called the *axis of rotation*. Hence there is a 3×3 orthognal matrix Q so that $T(\mathbf{x}) = Q\mathbf{x}$. Any vector $\mathbf{x}$ can be decomposed into the sum of its components in $\mathbf{x}_{\|}$ ℓ and $\mathbf{x}_{\perp}$ in $\ell^{\perp}$: $\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$. Since $\mathbf{x}_{\|}$ is fixed by Q,

$$\begin{aligned} Q\mathbf{x} &= Q\mathbf{x}_{\|} + Q\mathbf{x}_{\perp} \\ &= \mathbf{x}_{\|} + Q\mathbf{x}_{\perp} \ . \end{aligned}$$

$Q\mathbf{x}_{\perp}$ is what you get by rotating $\mathbf{x}_{\perp}$ in the plane $\ell^{\perp}$ through some angle θ that can always be chosen in the range $0 \leq \theta \leq \pi$ by rotating one way or the other.



We are now ready to make a connection between our algebraic definition of rotation matrices (5.1), and our geometric "picture" of what a rotation is. We have already explained that if a transformation T of $I\!R^3$ is a rotation, it is rigid, and therefore there is an orthogonal matrix Q so that $T(\mathbf{x}) = Q\mathbf{x}$ for all $\mathbf{x}$. This explains the $Q^tQ = I$ part of (5.1), but what about the $\det(Q) = 1$ part?

This can be explained by thinking in terms of the *rotation process* leading up to Q. For each t with $0 \leq t \leq \theta$, let $Q(t)$ denote the rotation about ℓ through the angle t in the same sense as Q. When $t = 0$, there is no rotation at all, and so $Q(0) = I$, the identity matrix. When $t = \theta$, we are rotating through the full angle θ, and so $Q(\theta) = Q$. For values of t in between, we are rotating by some angle in between. It is intuitively clear that as t increases from 0 to θ, the rotation $Q(t)$ *continuously* interpolates between I and Q – it describes a rotation process that builds Q up over time.

Now, for each t, Q is an orthogonal matrix and so $\det Q(t) = \pm 1$. Also, since the entries of $Q(t)$ depend continuously on t, so does $\det Q(t)$. After all, the determinant is just a sum of products of the entries. To see that $\det(Q) = +1$, note that $\det Q(0) = \det I = +1$. Now a continuous function of t cannot jump from $+1$ to -1. Since it *starts* at $+1$, it must *stay* at $+1$. Hence $\det Q(t) = +1$ for all t. In particular, $\det Q = 1$.

We now see that if a transformation T of $I\!R^3$ is a rotation, then $T(\mathbf{x}) = Q\mathbf{x}$ for some matrix Q satisfying (5.1). That is, every rotation in $I\!R^3$ is given by a matrix Q satisfying (5.1).

It remains to show the converse: That is, if Q is a matrix satisfying (5.1), and we define a transformation T of $I\!R^3$ by $T(\mathbf{x}) = Q\mathbf{x}$, then T is the rotation through some angle θ about some axis ℓ. This is a theorem, due to Euler, who proved it in 1752 in the course of an investigation of rigid body motion. In the next subsection, we explain why it is true, and how to compute θ and ℓ.

5.3: Euler's Theorem

We will prove Euler's Theorem in two steps. The first is a lemma that will give us the axis of rotation:

Lemma 1 *Let Q be any 3×3 matrix satisfying (5.1). Then 1 is an eigenvalue of Q. If $Q \neq I$, the corresponding eigenspace is exactly one dimensional.*

Proof: Let Q be any 3×3 rotation matrix. Then $\det(Q^t) = 1$, and so, since Q is invertible, with $Q^{-1} = Q^t$,

$$\begin{aligned} \det(Q - I) &= \det(Q^t)\det(Q - I) \\ &= \det(Q^tQ - Q^t) \\ &= \det((I - Q)^t) \\ &= \det((I - Q)) \ . \end{aligned}$$

But $(I - Q) = (-I)(Q - I)$ and $\det(-I) = (-1)^3 = -1$, so we conclude that

$$\det(Q - I) = -\det(Q - I) \ . \tag{5.4}$$

This means that $\det(Q - I) = 0$, and hence 1 is an eigenvalue of Q.

Since 1 is an eigenvalue of Q, the corresponding eigenspace is always *at least* one dimensional. Suppose it is at least two dimensional. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two orthonormal vectors in the eigenspace, and let $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a (right handed) orthonormal basis of $I\!R^3$. Then, by (5.2),

$$Q\mathbf{u}_3 = Q(\mathbf{u}_1 \times \mathbf{u}_2) = Q\mathbf{u}_1 \times Q\mathbf{u}_2 = \mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \ .$$

Therefore, $Q[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = [Q\mathbf{u}_1, Q\mathbf{u}_2, Q\mathbf{u}_3] = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. Letting $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, we have $QU = U$. Since U is invertible (it is a rotation), this means $Q = I$. ■

Let Q be any rotation with $Q \neq I$. Let $\mathbf{v}$ be any eigenvector of Q with eigenvalue 1. Consider the line ℓ given parametrically by $\mathbf{x}(t) = t\mathbf{v}$. This line is the eigenspace of Q corresponding to the eigenvalue 1. For any t, $Q(t\mathbf{v}) = tQ\mathbf{v} = t\mathbf{v}$, so points on this line are left unchanged by Q. This is the axis of rotation.

• *Finding the line ℓ fixed by Q is essentially an eigenvalue problem*

Definition Let Q be any 3×3 rotation matrix with $Q \neq I$. Then the line through the origin that is the eigenspace of Q with eigenvalue 1 is called the *axis of rotation* of Q. The plane through the origin orthogonal to this line is called the *plane of rotation*.

Example 1 (Finding the axis and plane of rotation) Let $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$. As you can easily check, Q is a rotation. To find the vectors $\mathbf{v}$ with $Q\mathbf{v} = \mathbf{v}$, we form

$$Q - I = \frac{1}{3}\begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ -2 & 2 & -4 \end{bmatrix} .$$

By what we just saw, this matrix has a non zero kernel. We could solve for it by row reduction, but this is a particularly simple case: The second column of $Q - I$ is the opposite of the first column, and so the kernel of $Q - I$ is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The line fixed by Q is the line through this vector and the origin; this is the axis of rotation of Q. Notice that the orthogonal complement to this line is the plane consisting of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y = 0$, and hence this is the equation for the plane of rotation.

Theorem 3 (Euler's Theorem) *Let Q be any 3×3 matrix satisfying (5.1). Then there is an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $I\!R^3$ so that in this basis, the matrix of the transformation induced by Q has the form*

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

for some angle θ.

The matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ clearly describes a rotation through an angle θ about the third axis. Now, the matrix in (5.5) represents the *same* transformation of $I\!R^3$ as does Q. It is different as a matrix because it describes this transformation using a different basis, but the transformation is the same. Hence Q also describes a rotation through an angle θ about the third axis in the basis; i.e., line along ℓ, which is the line through the origin and $\mathbf{u}_3$. Thus, Euler's Thoerem, as stated, does indeed imply that every 3×3 matrix Q satsifying (5.1) induces a roation of $I\!R^3$ through some angle θ about some axis ℓ.

Proof of Theorem 3: Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis of $I\!R^3$ in which $\mathbf{u}_3$ is an eigenvector of Q with eigenvalue 1. The action of Q on these basis vectors is particularly simple. First, since $Q\mathbf{u}_3 = \mathbf{u}_3$,

$$Q\mathbf{u}_1 \cdot \mathbf{u}_3 = Q\mathbf{u}_1 \cdot Q\mathbf{u}_3 = \mathbf{u}_1 \cdot \mathbf{u}_3 = 0$$

and likewise $Q\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$. Hence $Q\mathbf{u}_1$ and $Q\mathbf{u}_2$ are linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$ alone. That is, for some numbers a, b, c and d,

$$Q\mathbf{u}_1 = a\mathbf{u}_1 + b\mathbf{u}_2 \qquad \text{and} \qquad Q\mathbf{u}_2 = c\mathbf{u}_1 + d\mathbf{u}_2 \ , \tag{5.6}$$

Let $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. The matrix with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for the transormation of $I\!R^3$ induced by Q is the matrix $T = U^{-1}QU = U^tQU$.

From the fundamental formula for matrix multiplication,

$$\begin{aligned} QU &= Q[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] \\ &= [Q\mathbf{u}_1, Q\mathbf{u}_2, Q\mathbf{u}_3] \\ &= [a\mathbf{u}_1 + b\mathbf{u}_2, c\mathbf{u}_1 + d\mathbf{u}_2, \mathbf{u}_3] \\ &= [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] \begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \ . \end{aligned}$$

Hence, defining $T = U^tQU$, $T = \begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since T is a product of orthognal matrices, it is orthogonal, and so $T^tT = I$ Moreover, $\det(T) = \det(U^t)\det(Q)\det(U) = (\det(U))^2 = 1$, and so T is itself a rotation matrix.

Since the columns of T are othonormal, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ a unit vector in $I\!R^2$. Any unit vector in $I\!R^2$ can be written as $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ for some uniquely determined angle θ in $[0, 2\pi)$. Hence we can write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.

There are only two unit vectors in $I\!R^2$ orthogonal to $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, namely $\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$ and $-\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. It must be that $\begin{bmatrix} c \\ d \end{bmatrix}$ is one of these. Which one is it? With the first choice, $\det(T) = \cos^2(\theta) + \sin^2(\theta) = 1$, while with the second, we would have $\det(T) = -\cos^2(\theta) - \sin^2(\theta) = -1$, and this is impossible. Therefore $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$, and so

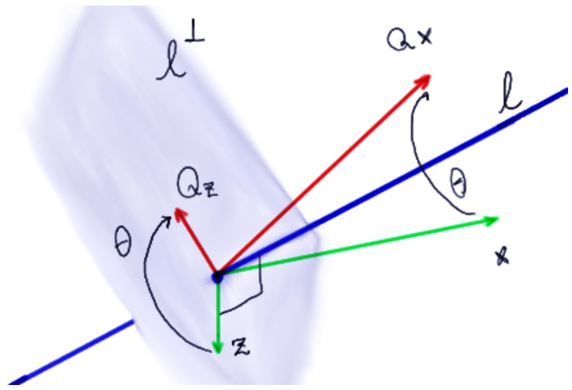
$$T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

■

We have now fully justified the use of the terminology "rotation matrix" for a 3×3 matrix Q satsifying (5.1). Every rotation is given by such a matrix, and the transformation of $I\!R^3$ induced by every such matrix is a rotation. In the next subsection, we will explain how to compute the angle of rotation.

5.4: The angle and direction of rotation

Consider again our diagram showing a rotation through an angle θ about an axis ℓ:



The vector $\mathbf{z}$ is carried into the vector $Q\mathbf{z}$ by a rotation through an angle θ that is clockwise when viewed as sown. However, the *same* result could have been obtained with a counterclockwise rotation through an angle $2\pi - \theta$. Which one shall we call *the* angle of rotation?

The standard convention is to define the angle of rotation θ so that it satisfies

$$0 \leq \theta \leq \pi \ .$$

One reason is that this makes θ easy to compute. Indeed, since Q is similar to the matrix

$$T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

and since similar matrices have the same traces, it follows that

$$\mathrm{tr}(Q) = \mathrm{tr}(T) = 1 + 2\cos(\theta) \ .$$

In other words,

$$\cos(\theta) = \frac{\mathrm{tr}(Q) - 1}{2} \ . \tag{5.7}$$

The fact that $\cos(\theta)$ is well defined is not in conflict with the fact that we need a convention for determining θ itself. Indeed, $\cos(\theta) = \cos(2\pi - \theta)$ for all θ. However, since the arccosine function $\cos^{-1}$ takes its values in $[0, \pi]$, the convention we have chosen gives us the formula:

$$\theta = \cos^{-1}\left(\frac{1 + \mathrm{tr}(Q)}{2}\right) \ . \tag{5.8}$$

Example 2 (Finding the angle of rotation) Let $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ as in Example 1. Evidently $\mathrm{tr}(Q) = 1/3$, so $(\mathrm{tr}(Q) - 1)/2 = -1/3$. Therefore, (5.7) gives

$$\theta = \arccos(-1/3) \ .$$

Next, there are two unit vectors in ℓ. In other words, there are two unit normal vectors to the plane of rotation. We will single one of these out and call it $\mathbf{u}$, the *direction of rotation.* As long as the angle of rotation is neither 0 nor π, the rotation will look clockwise when viewed from one side of the plane of rotation, and counterclockwise when viewed from the other. We define $\mathbf{u}$ so that it points to the side from which the rotation appears to be counterclockwise.

It is easy to give a formula for $\mathbf{u}$: Let $\mathbf{v}$ be any unit vector in the plane of rotation. Then $Q\mathbf{v}$ is another unit vector in the plane of rotation. By the right hand rule, $\mathbf{v} \times Q\mathbf{v}$ is a vector whose direction is orthogonal to the plane of rotation, and so that if you line your right hand up so that your thumb points along this direction, the fingers of your right hand curl in the direction of the rotation as it carries $\mathbf{v}$ into $Q\mathbf{v}$. As you can see, this is counterclockwise.

By the properties of the cross product, the magnitude of $\mathbf{v} \times Q\mathbf{v}$ is $\sin(\theta)$. If $\theta = 0$ or if $\theta = \pi$, then $\sin(\theta) = 0$, and the direction vector is not defined. Otherwise it is given by

$$\mathbf{u} = \frac{1}{\sin(\theta)} \mathbf{v} \times Q\mathbf{v} \ . \tag{5.9}$$

We can now give a "right hand rule" description of the rotation Q. If you grasp the axis of rotation with your right hand so that your extended thumb points in the direction of rotation $\mathbf{u}$, then curling your fingers gives the sense of the rotation; Q rotates by the angle θ about the axis of rotation ℓ in this sense. We see that $\mathbf{u}$ and θ are all the information we need to specify a rotation matrix Q. Once you know $\mathbf{u}_3$ and θ, you know, at least in geometric terms, how Q transforms any vector. In the next section, we shall see how to explicitly reconstruct Q from θ and $\mathbf{u}$.

Example 3 (Finding the direction of rotation) Let Q be the orthogonal matrix studied in Examples 1 and 2.

We found in Example 1 that the axis of rotation of Q is the line through $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and the origin. Clearly the vector $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to $\mathbf{w}$, and hence is a unit vector in the plane of rotation.

Hence we choose $\mathbf{v} = \mathbf{e}_3$. We compute $Q\mathbf{v} = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$, and $\mathbf{v} \times Q\mathbf{v} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Normalizing this vector, we find $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Exercises

In the Exercises 1.1 through 1.5, let Q_1, Q_2, Q_3 and Q_4 be the following rotation matrices:

$$Q_1 = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix} \qquad Q_2 = \frac{1}{75}\begin{bmatrix} -23 & -14 & 70 \\ -14 & 73 & 10 \\ -70 & -10 & -25 \end{bmatrix}$$

$$Q_3 = \frac{1}{45}\begin{bmatrix} 35 & -20 & -20 \\ 4 & 35 & -28 \\ 28 & 20 & 29 \end{bmatrix} \qquad Q_4 = \frac{1}{117}\begin{bmatrix} 77 & -68 & -56 \\ 4 & 77 & -88 \\ 88 & 56 & 53 \end{bmatrix} .$$

5.1 Check that Q_1 is indeed a rotation matrix. Then, find the angle of rotation θ and direction of rotation $\mathbf{u}$ for Q_1. Also, find an equation for the plane of rotation.

5.2 Check that Q_2 is indeed a rotation matrix. Then, find the angle of rotation θ and direction of rotation $\mathbf{u}$ for Q_2. Also, find an equation for the plane of rotation.

5.3 Check that Q_3 is indeed a rotation matrix. Then, find the angle of rotation θ and direction of rotation $\mathbf{u}$ for Q_3. Also, find an equation for the plane of rotation.

5.4 Check that Q_4 is indeed a rotation matrix. Then, find the angle of rotation θ and direction of rotation $\mathbf{u}$ for Q_4. Also, find an equation for the plane of rotation.

5.5 Let Q be any 3×3 rotation matrix. Explain why the cross product of any two non proportional rows of $Q - I$ is an eigenvector of Q with eigenvalues 1. Use this observation to compute the axis of rotation for Q_1 and Q_2 above.

5.6 (a) Using Maple, Matlab, or some such program, generate a random $3\times$ matrix A. Find a QR decomposition of A, and compute $\det(Q)$. If the result is $+1$, then Q is a rotation. Otherwise, $-Q$ is a rotation. Take the rotation generated this way, and compute the angle of rotation.

(b) Repeat the procedure in part **(a)** twenty times, and graph the results. does it look like the random angles are uniformly distributed over the interval $[0, \pi]$ or not?

5.7 Let θ be some angle in $[0, 2\pi]$, and let $\mathbf{u}$ be some unit vector in $I\!R^3$. Let $Q(\mathbf{u}, \theta)$ denote the rotation with this angle and direction of rotation. Explain why $Q(\theta, -\mathbf{u})$ is the inverse of $Q(\mathbf{u}, \theta)$.

5.8 Let θ be some angle in $[0, 2\pi]$, and let $\mathbf{u}$ be some unit vector in $I\!R^3$. Let $Q(\mathbf{u}, \theta)$ denote the rotation with this angle and direction of rotation. What are the angle and direction of rotation of $(Q(\mathbf{u}, \theta))^2$? How about $(Q(\mathbf{u}, \theta))^3$?

5.9 Let θ be some angle in $[0, 2\pi]$, and let $\mathbf{u}$ be some unit vector in $I\!R^3$. Let $Q(\mathbf{u}, \theta)$ denote the rotation with this angle and direction of rotation. Show that there is always a rotation matrix Q so that $Q^2 = Q(\mathbf{u}, \theta)$. That is, every rotation matrix has a square root that is also a rotation matrix.

5.10 Let $\mathbf{u}$ be any unit vector in $I\!R^3$. Let $M_{\mathbf{u}}$ denote the reflection matrix $I - 2\mathbf{u}\mathbf{u}^t$.

(a) Compute $\det(M_{\mathbf{u}})$.

(b) Show that for any two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $I\!R^3$, the product $M_{\mathbf{u}_1} M_{\mathbf{u}_2}$ is a rotation.

(c) What are the axis and angle of rotation of $M_{\mathbf{u}_1} M_{\mathbf{u}_2}$? Express your answer in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(d) Show that every 3×3 rotation matrix Q can be written as $Q = M_{\mathbf{u}_1} M_{\mathbf{u}_2}$ for some $\mathbf{u}_1$ and $\mathbf{u}_2$, and in fact, that there are infinitely many ways to do this.

5.11 Every 3×3 orthogonal matrix is either a reflection or a rotation. Is this true of 3×3 orthogonal matrices?

Section 6: The Generator of a Rotation and Interpolation

6.1 Euler's formula for the rotation matrix in terms of θ and $\mathbf{u}$.

We have seen in the previous section that every 3×3 rotation matrix Q with $Q^2 \neq I$ has a uniquely determined direction of rotation and and angle of rotation, and we have seen how to compute them given Q. We also know that the direction of rotation and and angle of rotation determine Q; there is just one rotation matrix with any given direction vector $\mathbf{u}$ and angle of rotation θ. Call this rotation matrix $Q(\mathbf{u}, \theta)$. We will now deduce a formula for $Q(\mathbf{u}, \theta)$.

First, consider a special case. Let $\mathbf{z}$ be any non zero vector in the plane of rotation. In this case,

$$Q\mathbf{z} = \cos(\theta)\mathbf{z} + \sin(\theta)(\mathbf{u} \times \mathbf{z}) \ . \tag{6.1}$$

To see this, observe that $\mathbf{u} \times \mathbf{z}$ is a vector of length $|\mathbf{z}|$ that is orthogonal to both $\mathbf{z}$ and $\mathbf{u}$. In particular, $\{\mathbf{z}, \mathbf{u} \times \mathbf{z}\}$ is a basis for the plane of rotation. Since $Q\mathbf{z}$ lies in the plane of rotation, we may write

$$Q\mathbf{z} = a\mathbf{z} + b(\mathbf{u} \times \mathbf{z})$$

for some a and b.

To determine a, take the dot product of both sides with $\mathbf{z}$. We get

$$a = \frac{\mathbf{z} \cdot Q\mathbf{z}}{|\mathbf{z}|^2} \ .$$

the right hand side is the cosine of the angle between $\mathbf{z}$ and $Q\mathbf{z}$, which is of course θ, the angle of rotation. That is, $a = \cos(\theta)$.

To determine b, take the cross product of both sides with $\mathbf{z}$. We get

$$\mathbf{z} \times Q\mathbf{z} = b\mathbf{z} \times (\mathbf{u} \times \mathbf{z}) \ .$$

Since $\mathbf{u}$ and $\mathbf{z}$ are orthogonal and $\mathbf{u}$ is a unit vector, the right hand rule gives us $\mathbf{z}\times(\mathbf{u}\times\mathbf{z}) = |\mathbf{z}|^2\mathbf{u}$. Hence with $\mathbf{v}$ defined by $\mathbf{v} = \dfrac{1}{|\mathbf{z}|}\mathbf{z}$,

$$\mathbf{v} \times Q\mathbf{v} = b\mathbf{u} \ .$$

By the fromula (5.9) for the direction vector, $b = \sin(\theta)$.

Our determination of the values of a and b verifies the formula (6.1). Now consider the general case. Let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^3$, and let $\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$ be the orthogonal decomposition of $\mathbf{x}$ into its components along the axis of rotation and in the plane of rotation. Then, since the orthogonal projection onto the axis of rotation is $\mathbf{u}\mathbf{u}^t$, we have

$$\mathbf{x}_{\|} = (\mathbf{u}\mathbf{u}^t)\mathbf{x}$$

and

$$\mathbf{x}_{\perp} = (I - \mathbf{u}\mathbf{u}^t)\mathbf{x} \ .$$

Then since

$$Q\mathbf{x} = Q\mathbf{x}_{\|} + Q\mathbf{x}_{\perp} = \mathbf{x}_{\|} + Q\mathbf{x}_{\perp} \ ,$$

and $Q\mathbf{x}_{\perp}$ is given by (6.1). We can simplify the result by noticing that since $\mathbf{u} \times \mathbf{x}_{\|} = 0$, we have $\mathbf{u} \times \mathbf{x}_{\perp} = \mathbf{u} \times \mathbf{x}$. This gives us Euler's formula for $Q = Q(\mathbf{u}, \theta)$:

$$Q(\mathbf{u}, \theta)\mathbf{x} = (\mathbf{u}\mathbf{u}^t)\mathbf{x} + \cos(\theta)(I - \mathbf{u}\mathbf{u}^t)\mathbf{x} + \sin(\theta)(\mathbf{u} \times \mathbf{x}) \ . \tag{6.2}$$

To write this down in matrix form, we need to express the final term in matrix form; the first two already are. Here is how: Consider the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$\mathbf{x} \quad \to \quad \mathbf{u} \times \mathbf{x} \ .$$

If we write $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have that

$$\mathbf{u} \times \mathbf{x} = \begin{bmatrix} bz - cy \\ cx - az \\ ay - bx \end{bmatrix} \ . \tag{6.3}$$

Each entry on the right hand side is a linear function of x, y and z, so this is a linear transformation. Therefore, we can find a matrix $B_{\mathbf{u}}$ so that

$$B_{\mathbf{u}}\mathbf{x} = \mathbf{u} \times \mathbf{x} \ .$$

The jth column of $B_{\mathbf{u}}$ will be $B_{\mathbf{u}}\mathbf{e}_j$, and from (6.3) we have

$$\mathbf{u} \times \mathbf{e}_1 = \begin{bmatrix} 0 \\ c \\ -b \end{bmatrix} \qquad \mathbf{u} \times \mathbf{e}_2 = \begin{bmatrix} -c \\ 0 \\ a \end{bmatrix} \qquad \mathbf{u} \times \mathbf{e}_3 = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} \ .$$

Therefore,

$$B_{\mathbf{u}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \ . \tag{6.4}$$

As you can easily check,

$$B_{\mathbf{u}}\mathbf{x} = \mathbf{u} \times \mathbf{x} \ .$$

We can now write (6.2) in matrix form as

$$Q(\mathbf{u}, \theta) = \cos(\theta)I + (1 - \cos(\theta))\mathbf{u}\mathbf{u}^t + \sin(\theta)B_{\mathbf{u}} \ . \tag{6.5}$$

Example 1 (Computing $Q(\mathbf{u}, \theta)$) Let $\mathbf{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and let $\theta = \arccos(5/13)$. We will now use (6.5) to compute $Q(\mathbf{u}, \theta)$.

First,

$$\sin(\arccos(5/13)) = 12/13 \ .$$

Next,

$$\mathbf{u}\mathbf{u}_t = \frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \ ,$$

and finally,

$$B_{\mathbf{u}} = \frac{1}{9}\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \ .$$

Putting it all together,

$$\begin{aligned} Q(\mathbf{u},\theta) &= \frac{5}{13}I + \frac{8}{13}\frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \frac{12}{13}\frac{1}{3}\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \\ &= \frac{1}{117}\begin{bmatrix} 77 & -4 & 88 \\ 68 & 77 & -56 \\ -56 & 88 & 53 \end{bmatrix} \ . \end{aligned}$$

6.2 Using Euler's formula to compute u.

We learned earlier that every 3×3 matrix Q satisfying the algebraic relations $Q^tQ = I$ and $\det(Q) = 1$ is, considered geometrically as a transformation of $I\!R^3$, a rotation through some angle θ about some axis along the direction $\mathbf{u}$.

Using direct geometric reasoning, we deduced Euler's formula (6.5) for Q in terms of θ and $\mathbf{u}$. We can also use this formula to easily compute θ and $\mathbf{u}$ for any rotation matrix Q. Here is how:

The first two terms in (6.5) are symmetric matrices. Indeed, $(\mathbf{u}\mathbf{u}^t)^t = (\mathbf{u}^t)^t\mathbf{u}^t = \mathbf{u}\mathbf{u}^t$. The last term is antisymmetric since clearly $(B_{\mathbf{u}})^t = -B_{\mathbf{u}}$. Now let Q be any 3×3 matrix satisfying $Q^tQ = I$ and $\det(Q) = 1$. Then we know that for some θ and $\mathbf{u}$,

$$Q = \cos(\theta)I + (1 - \cos(\theta))\mathbf{u}\mathbf{u}^t + \sin(\theta)B_{\mathbf{u}} \ . \tag{6.6}$$

Taking the transpose,

$$Q^t = \cos(\theta)I + (1 - \cos(\theta))\mathbf{u}\mathbf{u}^t - \sin(\theta)B_{\mathbf{u}} \ . \tag{6.7}$$

Therefore, subtracting (6.7) from (6.6) and dividing by 2,

$$\sin(\theta)B_{\mathbf{u}} = \frac{1}{2}\left(Q - Q^t\right) \ . \tag{6.8}$$

Since we already have an explicit formula for θ in terms of Q, namely $\theta = \cos^{-1}(\mathrm{tr}(Q) - 1)/2)$, this gives us an explicit formula for $\mathbf{u}$, and by computing

$$\frac{1}{2}\left(Q - Q^t\right) \qquad \text{and} \qquad \mathrm{Tr}(Q) \ ,$$

we can determine both $\mathbf{u}$ and θ.

Example 2 (Computing u and θ)) Let $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$. As you can easily check, Q is a rotation. We have,

$$\frac{1}{2}\left(Q - Q^t\right) = \frac{1}{6}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix} .$$

Now from (6.4) and (6.7), we have

$$\sin(\theta)\mathbf{u} = \frac{1}{6}\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} .$$

Since $\sin(\theta)$ is positive for $0 < \theta < \pi$, he length of the vector on the right must be $\sin(\theta)$. We can eliminate $\sin(\theta)$ by dividing by the length. doing so, we find

$$\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} .$$

since the trace of Q is $1/3$, we have $\theta = \cos^{-1}(1/3)$.

Let's do one more. This time, let $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix}$. As you can check, this is a rotation matrix. Computing,

$$\frac{1}{2}(Q - Q^t) = \frac{1}{9}\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} .$$

From this it follows that

$$\sin(\theta)\mathbf{u} = \frac{1}{9}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} = \frac{\sqrt{17}}{9}\left(\frac{1}{\sqrt{17}}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix}\right) .$$

and hence

$$\mathbf{u} = \frac{1}{\sqrt{17}}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} .$$

Next, we find that $\mathrm{Tr}(Q) = -7/9$. This means that $\cos(\theta) = (-7/9 - 1)/2 = -8/9$, and so

$$\theta = \cos^{-1}(-8/9) .$$

Now let's pause to take stock of what we have learned. We have two ways to think about 3×3 rotation matrices: We can think of them algebraically, in terms of the relations $Q^tQ = I$ and $\det(Q) = 1$. Alternatively, we can think about them geometrically in terms of their axis and angle of rotation. We know how to go back and forth between the two points of view. One reason this is useful is that some things are easier to see algebraically, and others are easier to see geometrically.

For example, Theorem 1.1 says that the product of two rotations is again a rotation. This had an easy algebraic proof. It is probably less easy to see this directly in geoemtric terms. Indeed, if you reflect $I\!R^3$ about two planes, the result is *not* another reflection. Why should it be that if we rotation $I\!R^3$ through the angle θ_1 about $\mathbf{u}_1$, and then rotate it through the angle θ_2 about $\mathbf{u}_2$, the combined result is the same as rotating $I\!R^3$ through some angle θ about some unit vector $\mathbf{u}$?

If you think about this question, you will probably find that it is not easy to visualize the new angle and axis. Nonetheless, we now have the means to compute them: Let Q_1 and Q_2 be the individual rotations. Compute the matrix product $Q = Q_1Q_2$. By Theorem 1.1, this is again a rotation. We can find the corresponding θ and $\mathbf{u}$ using (6.7).

Example 3 (Computing u and θ) for a product of rotations) Let $Q_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ and let $Q_2 = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix}$. These are the rotation matrices from Example 2. The product $Q = Q_1Q_2$ is also a rotation matrix. What are the angle θ and direction $\mathbf{u}$ of rotation for Q?

To find this, we first compute that

$$Q = Q_1Q_2 = \frac{1}{27}\begin{bmatrix} 25 & 2 & 10 \\ -10 & 10 & 23 \\ -2 & -25 & 10 \end{bmatrix},$$

and from this that

$$\frac{1}{2}(Q - Q^t) = \frac{1}{9}\begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 8 \\ -2 & -8 & 0 \end{bmatrix}.$$

From this we see that

$$\sin(\theta)\mathbf{u} = \frac{1}{9}\begin{bmatrix} -8 \\ 2 \\ -2 \end{bmatrix}.$$

The length of $\begin{bmatrix} -8 \\ 2 \\ -2 \end{bmatrix}$ is $6\sqrt{2}$ so that

$$\begin{bmatrix} -8 \\ 2 \\ -2 \end{bmatrix} = 6\sqrt{2}\left(\frac{1}{6\sqrt{2}}\begin{bmatrix} -8 \\ 2 \\ -2 \end{bmatrix}\right)$$

Also, we note that $\mathrm{Tr}(Q) = 45/27$. Hence $\cos(\theta) = (45/27 - 1)/2 = 1/3$,

$$\theta = \cos^{-1}(1/3) \qquad \text{and} \qquad \mathbf{u} = \frac{1}{3\sqrt{2}}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

6.3 Application to interpolation

What are the practical problems that motivate our study of 3×3 rotation matrices? There are many. One important one is the *interpolation problem.*

To explain this, consider the picture at the beginning of this section. Let $\mathbf{x}_0$ be the location of the reference point at time $t = 0$, and let Q_0 be the rotation matrix given by $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ at time $t = 0$. Likewise, let $\mathbf{x}_1$ be the location of the reference point at time $t = 0$, and let Q_1 be the rotation matrix given by $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ at time $t = 1$.

If we want to make an animation showing the motion between times $t = 0$ and $t = 1$, we need to find a continuous time dependent vector $\mathbf{x}(t)$ with

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad \text{and} \qquad \mathbf{x}(1) = \mathbf{x}_1 \tag{6.9}$$

and a continuous time dependent rotation matrix $Q(t)$ with

$$Q(0) = Q_0 \qquad \text{and} \qquad Q(1) = Q_1 \ . \tag{6.10}$$

It is easy to find an interpolation satisfying (6.9): We can use the straight line segment connecting $\mathbf{x}_0$ and $\mathbf{x}_1$:

$$\mathbf{x}(t) = (1-t)\mathbf{x}_0 + t\mathbf{x}_1 = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \ .$$

As you can easily check, this satisfies the requirement (6.9).

It is not so easy to interpolate between the rotation matrices Q_0 and Q_1. We cannot simply use

$$(1-t)Q_0 + tQ_1$$

since this generally will not be a rotation for $0 < t < 1$. For example, let $Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. You can easily check that these are both rotations. However

$$(1-t)Q_0 + tQ_1 = \begin{bmatrix} 1-2t & 0 & 0 \\ 0 & 1-2t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and this does not have orthonormal columns for $0 < t < 1$, so while (6.10) is satisfied, the interpolating matrices $Q(t)$ are not rotations, and hence cannot be used to describe the configuration of our rigid body at time t.

In two dimensions, it is easy to see how to interpolate between to rotations *using* rotations. As we have seen, the general 2×2 rotation matrix has the form

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{6.11}$$

where θ is some angle in the interval$[0, 2\pi)$.

To interpolate between $Q_0 = \begin{bmatrix} \cos(\theta_0) & -\sin(\theta_0) \\ \sin(\theta_0) & \cos(\theta_0) \end{bmatrix}$ and $Q_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$, just interpolate the angle: Define

$$\theta(t) = (1-t)\theta_0 + t\theta_1 \ .$$

Then, as you can easily see,

$$Q(t) = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix}$$

satisifies (6.10), and *is* a 2×2 rotation matrix for each t.

To find such an interpolation for 3×3 rotation matrices, we need to find the three dimensional analog of (6.11):

Define

$$U = Q_1 Q_0^t \ .$$

Since this is a product of rotations, it is itself a rotation, Therefore, we can write it in the form $U = Q(\mathbf{u}, \theta)$ for some unit vector $\mathbf{u}$ and some angle θ with $0 < \theta \leq \pi$. (Since $Q_0 \neq Q_1$, $U \neq I$, and $\theta \neq 0$).

Now for $0 \leq t \leq 1$, define the rotation $U(t)$ by

$$U(t) = Q(\mathbf{u}, t\theta) \ .$$

Then define $Q(t)$ by

$$Q(t) = U(t) Q_0 \ .$$

Since for each t this is a product of rotations, it is itself a rotation. This does the trick. Since $U(0) = I$,

$$Q(0) = I Q_0 = Q_0 \ .$$

Also, since $U(1) = Q = Q_1 Q_0^t$,

$$Q(1) = Q_1 Q_0^t Q_0 = Q_1 I = Q_1 \ .$$

This gives us our interpolation.

To interpolate between Q_0 and Q_1 we need to be able to compute the matrix $Q(\mathbf{u}, t\theta)$ for $0 < t < 1$ where $Q(\mathbf{u}, \theta) = Q_1 Q_0^t$.* What we have just learned makes this easy.

Example 4 (Computing an interpolation)) Let $Q_0 = \frac{1}{9}\begin{bmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ 4 & 7 & -4 \end{bmatrix}$ and $Q_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$. Then, recognizing these matrices from Example 3, we have thatt $Q_1 Q_0^t = Q(\mathbf{u}, \theta)$ where

$$Q(\mathbf{u}, \theta) = \frac{1}{27}\begin{bmatrix} 25 & 2 & 10 \\ -10 & 10 & 23 \\ -2 & -25 & 10 \end{bmatrix} \ .$$

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* Note what we are *not* doing in our iinterpolation between Q_0 and Q_1: We are *not* making a linear interpolation between $(\mathbf{u}_0, \theta_0)$ and $(\mathbf{u}_1, \theta_1)$. One could define

$$\mathbf{u}(t) = \frac{(1-t)\mathbf{u}_0 + t\mathbf{u}_1}{|(1-t)\mathbf{u}_0 + t\mathbf{u}_1|} \qquad \text{and} \qquad \theta(t) = (1-t)\theta_0 + t\theta_1 \ .$$

Then we *could* interpolate between $Q(\mathbf{u}_0, \theta_0)$ and $Q(\mathbf{u}_1, \theta_1)$ using

$$Q(t) = Q(\mathbf{u}(t), \theta(t)) \ .$$

This is *not* what we do. Our interpolation is different: if is based on consideration of $Q_1 Q_0^t$. In the next section, we shall see why this is the "most economical" interpolation, with the minimum of "twisting and turning".

In Example 3 we computed that

$$\theta = \cos^{-1}(1/3) \qquad \text{and} \qquad \mathbf{u} = \frac{1}{3\sqrt{2}}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} .$$

Hence with this value of θ and this value of $\mathbf{u}$, we define

$$\theta(t) = t\cos^{-1}(1/3)$$

and then define $U(t)$ using (6.5):

$$U(t) = \cos(\theta(t))I + (1 - \cos(\theta(t)))\mathbf{u}\mathbf{u}^t + \sin(\theta(t))B_{\mathbf{u}} .$$

Since

$$\mathbf{u}\mathbf{u}^t = \frac{1}{18}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad B_{\mathbf{u}} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} .$$

Hence, a bit more explicitly,

$$U(t) = \cos(\theta(t))\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1 - \cos(\theta(t)))\frac{1}{18}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} + \sin(\theta(t))\frac{1}{3\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} .$$

Finally, the interpolation is given by

$$Q(t) = U(t)Q_0 .$$

6.4 Rotations and matrix exponentials.

There is an identity relating $\mathbf{u} \times \mathbf{u}^t$ and $B_{\mathbf{u}}$ that opens the way to a cleaner expression of (6.5). The identity is

$$(B_{\mathbf{u}})^2 = (\mathbf{u}\mathbf{u}^t - I) . \qquad (6.12)$$

Indeed, let $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be any unit vector in $I\!R^3$. Then $B_{\mathbf{u}} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$ so that

$$B_{\mathbf{u}}^2 = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = \begin{bmatrix} -(b^2 + c^2) & ab & ac \\ ba & -(a^2 + c^2) & bc \\ ca & cb & -(a^2 + b^2) \end{bmatrix} .$$

Since $a^2 + b^2 + c^2 = 1$, this means

$$B_{\mathbf{u}}^2 = \begin{bmatrix} a^2 - 1 & ab & ac \\ ba & b^2 - 1 & bc \\ ca & cb & c^2 - 1 \end{bmatrix} = \mathbf{u}\mathbf{u}^t - I .$$

This identity allows us to simplify (6.5) and to express it just in terms of powers of $B_{\mathbf{u}}$:

$$\begin{aligned} Q(\mathbf{u},\theta) &= \cos(\theta)I + (1-\cos(\theta))\mathbf{u}\mathbf{u}^t + \sin(\theta)B_{\mathbf{u}} \\ &= I + (\cos(\theta)-1)I + (1-\cos(\theta))\mathbf{u}\mathbf{u}^t + \sin(\theta)B_{\mathbf{u}} \\ &= I + \sin(\theta)B_{\mathbf{u}} - (1-\cos(\theta))(B_{\mathbf{u}})^2 \ . \end{aligned} \tag{6.13}$$

There is another even simpler identity realting $B\mathbf{u}$ and $\mathbf{u}\mathbf{u}^t$, namely $B_{\mathbf{u}}\mathbf{u}\mathbf{u}^t = 0$. This follows from $B_{\mathbf{u}}\mathbf{u} = 0$. Using this and (6.12), it is simple to deduce the formulas

$$(B_{\mathbf{u}})^{2k} = (-1)^{k-1}B_{\mathbf{u}}^2 \tag{6.14}$$

and for all $k \geq 0$,

$$(B_{\mathbf{u}})^{2k+1} = (-1)^{k-1}B_{\mathbf{u}} \ . \tag{6.15}$$

We can apply these to see that

$$I + \sin(\theta)B_{\mathbf{u}} - (1-\cos(\theta))(B_{\mathbf{u}})^2 = e^{\theta B_{\mathbf{u}}} = e^{\theta B_{\mathbf{u}}} \ .$$

Indeed, by (6.15),

$$\begin{aligned} \sin(\theta)B_u &= \left(\sum_{k=0}^{\infty} \frac{\theta^{2k+1}(-1)^k}{(2k+1)!}\right) B_{\mathbf{u}} \\ &= \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!}\left((-1)^k B_{\mathbf{u}}\right) \\ &= \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!}(B_{\mathbf{u}})^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(\theta B_{\mathbf{u}})^{2k+1}}{(2k+1)!} \end{aligned}$$

Using (6.14), we have that

$$\begin{aligned} -(1-\cos(\theta))(B_{\mathbf{u}})^2 &= \sum_{k=1}^{\infty} \frac{\theta^{2k}}{(2k)!}(-1)^{k-1}(B_{\mathbf{u}})^2 \\ &= \sum_{k=1}^{\infty} \frac{\theta^{2k}}{(2k)!}\left(B_{\mathbf{u}}^{2k}\right) \ . \end{aligned}$$

Combining these last two formulas with (6.13), we have that

$$\begin{aligned} Q(\mathbf{u},\theta) &= I + \sum_{k=0}^{\infty} \frac{(\theta B_{\mathbf{u}})^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} \frac{\theta^{2k}}{(2k)!} B_{\mathbf{u}}^{2k} \\ &= \sum_{\ell=0}^{\infty} \frac{\theta^{\ell}\,(B_{\mathbf{u}})^{\ell}}{\ell!} \\ &= e^{\theta B_{\mathbf{u}}} \ . \end{aligned}$$

We have just shown that any rotation matrix Q is the exponential of an antisymmetric matrix B. There is a valid converse:

- *A 3 matrix Q is a rotation matrix if and only if $Q = e^tB$ where B is antisymetric.*

Indeed, we have already seen that any rotation matrix Q can be written as

$$Q = Q(\mathbf{u}, \theta) = e^{\theta B_{\mathbf{u}}} ,$$

where $\theta B_{\mathbf{u}}$ is antisymmetric.

To prove the converse, let $Q = e^B$ where B is antisymetric. Then $B^t = -B$, and so B and B^t certainly commute; i.e., $B^tB = BB^t$. This has the following significance here: If C is any 3×3 matrix such that $BC = CB$, so that the order of multiplication is not important in products of B and C, then

$$e^{B+C} = e^B e^C ,$$

just as if B and C were numbers. If $BC \neq CB$, things are completely different.

To apply this here, take $C = B^t = -B$. Then,

$$e^{B^t} e^B = e^{B^t + B} = e^0 = I .$$

It also follows from the power series representation that

$$(e^B)^t = e^{B^t} ,$$

and therefore, if we define $Q = e^B$, we have

$$Q^t Q = I .$$

That is, the exponential of an antisymmetric matrix is an isometry.

Better yet, it is actually a rotation. To see this, observe that since $(1/2)B$ commutes with itself,

$$e^B = e^{(1/2)B + (1/2)B} = e^{(1/2)B} e^{(1/2)B} .$$

Since $(1/2)B$ is antisymmetric, $e^{(1/2)B}$ is an isometry, and so $\det\left(e^{(1/2)B}\right) = \pm 1$. Either way,

$$\det\left(e^B\right) = \det\left(e^{(1/2)B}\right) \det\left(e^{(1/2)B}\right) = 1 .$$

We have proved the following:

Theorem 1 (Rotation matrices and antisymmetric matrices) *A 3×3 matrix Q is a rotation matrix if and only if there is a 3×3 antisymmetric matrix B so that $Q = e^B$. Moreover, if $\mathbf{u}$ is any unit vector and θ is any angle with $0 \leq \theta \leq \pi$, then the rotation $Q(\mathbf{u}, \theta)$ with direction of rotation $\mathbf{u}$ and angle of rotation θ is given by*

$$\begin{aligned} Q(\mathbf{u}, \theta) &= e^{\theta B_{\mathbf{u}}} \\ &= \cos(\theta) I + (1 - \cos(\theta)) \mathbf{u}\mathbf{u}^t + \sin(\theta) B_{\mathbf{u}} . \end{aligned}$$

It is natural to refer to a matrix B such that $e^B = Q$ a *logarithm* of Q.

Example 5 (Finding a logarithm of a rotation) Let Let Q be the orthogonal matrix studied in Examples 1, 2 and 3 of the previous section. We found that for this Q, $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\theta = \arccos(-1/3)$. Then

$$B_{\mathbf{u}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0 & -1\\0 & 0 & 1\\1 & -1 & 0\end{bmatrix} ,$$

and so

$$B = \theta B_{\mathbf{u}} = \frac{\arccos(-1/3)}{\sqrt{2}}\begin{bmatrix}0 & 0 & -1\\0 & 0 & 1\\1 & -1 & 0\end{bmatrix} .$$

Now consider $e^{tB_{\mathbf{u}}}$ for arbitrary values of t. We can write any number t in the form

$$t = n\pi + s$$

where n is an integer and s is in the interval $[0, \pi)$, and both n and s are uniquely determined. Since

$$\cos(n\pi + s) = (-1)^n \cos(s) \qquad \text{and} \qquad \sin(n\pi + s) = (-1)^n \sin(s) ,$$

$$e^{tB_{\mathbf{u}}} = e^{s((-1)^n B_{\mathbf{u}})} .$$

It follows that s is the angle of rotation of $e^{tB_{\mathbf{u}}}$, and the direction of rotation of $e^{tB_{\mathbf{u}}}$ is $(-1)^n\mathbf{u}$.

6.5 Angular velocity in $I\!R^3$

Let $Q(s)$ be a time dependent 3×3 rotation matrix, and suppose that $Q(s)$ can be differentiated entry by entry. Let $Q'(s)$ denote this derivative. Now, since the transpose of $Q(s)$ is also a rotation, $Q(s)Q^t(s) = I$. The right hand side is constant, and so

$$(Q(s)Q^t(s))' = I' = 0 .$$

By the product rule, $(Q(s)Q^t(s)) = Q'(s)Q^t(s) + Q(s)(Q^t(s))'$. But clearly, $(Q^t(s))' = (Q'(s))^t$, and so

$$\begin{aligned} Q'(s)Q^t(s) &= -Q(s)(Q'(s))^t \\ &= -(Q'(s)Q^t(s))^t . \end{aligned}$$

In other words, for each s,

$$A(s) = Q'(s)Q^t(s) \tag{6.16}$$

is an antisymmetric matrix.

We can rewrite (6.16) as

$$Q'(s) = A(s)Q(s) . \tag{6.17}$$

Writing the antisymmetric matrix $A(s)$ in the form

$$A(s) = \begin{bmatrix} 0 & -c(s) & b(s) \\ c(s) & 0 & -a(s) \\ -b(s) & a(s) & 0 \end{bmatrix} , \tag{6.18}$$

we define the corresponding vector $\mathbf{a}(s)$ by

$$\mathbf{a}(s) = \begin{bmatrix} a(s) \\ b(s) \\ c(s) \end{bmatrix} .$$

This vector $\mathbf{a}(s)$ is called the angular velocity vector of $Q(s)$:

Definition (Angular velocity) If $Q(s)$ is a differential 3×3 matric valued function of s, the corresponding angular velocity vector $\mathbf{a(s)}$ is the $I\!R^3$ values function $\mathbf{a}(s) = \begin{bmatrix} a(s) \\ b(s) \\ c(s) \end{bmatrix}$ such that the corresponding antisymmetric matrix $A(s)$ given by (6.18) satisfies (6.16).

Just as a path $\mathbf{x}(s)$ in $I\!R^3$ can be recovered from a specification of the velocity $\mathbf{x}'(s)$ and the starting position $\mathbf{x}_0$, a path in the space of rotations can be recovered from a specification of the angular velocity vector $\mathbf{a}(s)$ and the starting rotation Q_0.

If the angular velocity is constant; i.e., if $\mathbf{a}(s)$ and hence $A(s)$ do not actually depend on s, this can be done in closed form. Let A denote the constant antisymteric matrix A corresponding to the angular velocity. Then

$$Q(s) = e^{sA} Q_0 . \tag{6.19}$$

You can easily check that

$$Q'(s) = Ae^{sA} Q_0 = AQ(s) ,$$

so that this is one solution of (6.17), and this solution satisfies $Q(0) = Q_0$.

Now suppose that $\tilde{Q}(s)$ is any other such solution. Then

$$\begin{aligned} ((\tilde{Q}(s))^t Q(s))' &= ((\tilde{Q}(s))')^t Q(s) + (\tilde{Q}(s))^t Q'(s) \\ &= (A\tilde{Q}(s))^t Q(s) + (\tilde{Q}(s))^t AQ(s) \\ &= (\tilde{Q}(s))^t A^t Q(s) + (\tilde{Q}(s))^t AQ(s) \\ &= (\tilde{Q}(s))^t (A^t + A) Q(s) \\ &= 0 \end{aligned}$$

since $A^t = -A$. Therefore, $(\tilde{Q}(s))^t Q(s)$ is constant. Since $(\tilde{Q}(0))^t Q(0) = Q_0^t Q_0 = I$,

$$(\tilde{Q}(s))^t Q(s) = I$$

for all s, which means that $\tilde{Q}(s) = Q(s)$ for all s. That is, (6.19) is *the only* solution of (6.17) in this case.

In fact, the same argument shows that solutions to (6.17) are always unqiue whenever they exist. What having A constant helps with is the existence of any solution; only in this case do we have the nice formula (6.19).

If $A(s)$ really does depend on s, things are more subtle, but still (6.19) helps. A reasonable thing to do is to approximate $A(s)$ be a piecewise constant function, that is constant on time intervals of some step size $h > 0$.

For example, fix a step size $h > 0$. For each integer m, define $A_m = A(mh)$. We will approximate $A(s)$ by A_m in the time interval

$$mh \le s \le m+1)h \ .$$

The approximation is exact at the left hand side of the interval, and if h is small and the motion is smooth, it will still be good at the right side. Then, for s in the interval $0 \le s \le h$, define

$$Q(s) = e^{sA_0}Q_0 \ .$$

Next, for s in the interval $h \le s \le 2h$, define

$$Q(s) = e^{(s-h)A_1}Q(h) \ .$$

Next, for s in the interval $2h \le s \le 3h$, define

$$Q(s) = e^{(s-2h)A_2}Q(2h) \ ,$$

and so on.

This gives us an approximate solution of (6.19), and it may be shown that if $A(s)$ depends continuously on s, that this tends to the exact solution as h tends to zero. If $A(s)$ is differentiable, one gets a pretty good approximation even with a reasonable value of the step size.

Finally, we return once more to the interpolation problem. Recall that our recipe for interpolating between Q_0 and Q_1 is to form the matrix $Q_1Q_0^t$, and to write it in the form

$$Q_1Q_0^t = e^{\theta B_{\mathbf{u}}} \ .$$

We then define

$$Q(s) = e^{s\theta B_{\mathbf{u}}}Q_0 \ .$$

What is the angular velocity along this interpolation path? Thats easy:

$$(Q(s))' = \theta B_{\mathbf{u}} e^{s\theta B_{\mathbf{u}}}Q_0 = \theta B_{\mathbf{u}} Q(s)$$

which means that

$$A(s) = \theta B_{\mathbf{u}} \ .$$

This does not depend on s. In other words, our interpolation produces a path in the space of rotations that has constant angular velocity, just as the linear interpolation $(1-s)\mathbf{x}_0 + s\mathbf{x}_1$ between two points in $I\!R^3$ produces a path of constant velocity.

Moreover, the straight line segment $(1-s)\mathbf{x}_0+s\mathbf{x}_1$ is the shortest path in $I\!R^3$ connecting $\mathbf{x}_0$ and $\mathbf{x}_1$. In the next section we will see that our interpolation between Q_0 and Q_1 is the shortest path in the space of rotations that runs from Q_0 to Q_1. The first thing we will have to do there is to explain how one measures distance in the space of rotations.

Exercises

In the Exercises 2.1 through 2.8, let Q_1, Q_2, Q_3 and Q_4 be the following rotation matrices:

$$Q_1 = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix} \qquad Q_2 = \frac{1}{75}\begin{bmatrix} -23 & -14 & 70 \\ -14 & 73 & 10 \\ -70 & -10 & -25 \end{bmatrix}$$

$$Q_3 = \frac{1}{45}\begin{bmatrix} 35 & -20 & -20 \\ 4 & 35 & -28 \\ 28 & 20 & 29 \end{bmatrix} \qquad Q_4 = \frac{1}{117}\begin{bmatrix} 77 & -68 & -56 \\ 4 & 77 & -88 \\ 88 & 56 & 53 \end{bmatrix}.$$

6.1 Find the time dependent orthogonal matrix $U(t)$, $0 \le t \le 1$ so that $Q(t) = U(t)Q_1$ interpolates between Q_1 and Q_2.
6.2 Find the time dependent orthogonal matrix $U(t)$, $0 \le t \le 1$ so that $Q(t) = U(t)Q_1$ interpolates between Q_1 and Q_3.
6.3 Find the time dependent orthogonal matrix $U(t)$, $0 \le t \le 1$ so that $Q(t) = U(t)Q_1$ interpolates between Q_1 and $(Q_1)^2$.
6.4 Find the time dependent orthogonal matrix $U(t)$, $0 \le t \le 1$ so that $Q(t) = U(t)Q_1$ interpolates between Q_1 and $(Q_1)^3$.

6.5 Find all antisymetric matrices B so that $e^tB = Q_1$.

6.6 Find all antisymetric matrices B so that $e^tB = Q_2$.

6.7 Find all antisymetric matrices B so that $e^tB = Q_3$.

6.8 Find all antisymetric matrices B so that $e^tB = Q_4$.

6.9 Let $\mathbf{u}$ be any unit vector in $I\!R^3$, and let $B_\mathbf{u}$ be the corresponding antisymmetric matrix. Let Q be any rotation matrix. How are $QB_\mathbf{u}Q^t$ and $Q\mathbf{u}$ related?