Math 2605, **Exam#1**, 23 Sept 2010, **NAME:** SOLUTIONS

page	1	2	3	4	5	6	total
points							
maximum	20	20	20	20	20	15	

Neither cheatsheets nor calculators are allowed.

1. (20 points) Find a parametric representation of the line given by the system of equations:

$$\begin{cases} 2x - y + 3z = 4 \\ x + y + z = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x+y+z=2 \\ -3y+z=0 \end{cases} \Leftrightarrow \begin{array}{l} x = 2 - \frac{4}{3}z \\ y = \frac{1}{3}z \end{array} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Answer: $\vec{x}(t) = \underbrace{[2,0,0]^T}_{x_0} + t\,\underbrace{[-4,1,3]^T}_{v}$

Find the distance from $\boldsymbol{p} = (-1,-1,2)^T$ to the line.

$$w = p - x_0 = [-3,-1,2]^T$$

$$\|v\| = \sqrt{16+1+9} = \sqrt{26}$$

$$u = \frac{v}{\|v\|} = \frac{[-4,1,3]^T}{\sqrt{26}}$$

$$w_{\parallel} = (w \cdot u)\,u = \frac{12-1+6}{(\sqrt{26})^2}[-4,1,3]^T = \frac{17}{26}[-4,1,3]^T$$

$$w_{\perp} = \left[-3 + \frac{17\cdot 4}{26}, -1 - \frac{17}{26}, 2 - \frac{17\cdot 3}{26}\right] = \frac{1}{26}[10,-43,1]$$

$$\|w_{\perp}\| = \frac{1}{26}\sqrt{100+1849+1} = \frac{1}{26}\sqrt{1950}$$

2. (20 points) Consider two lines

$$x(s) = (s-1, 2s+1, s)^T \text{ and } y(t) = (1-t, 2+2t, 1+t)^T.$$

Find s and t such that $||x(s) - y(t)||$ is minimal.

"Calculus" solution:

let $f(s,t) = ||x(s) - y(t)||^2 = (s+t-2)^2 + (2s-2t-1)^2 + (s-t-1)^2$

$f_s = 2(s+t-2) + 4(2s-2t-1) + 2(s-t-1) = 12s - 8t - 2 = 0$

$f_t = 2(s+t-2) - 4(2s-2t-1) - 2(s-t-1) = -8s + 12t + 10 = 0$

Solve $\begin{cases} 6s - 4t = 1 \\ -4s + 6t = -5 \end{cases} \Leftrightarrow \begin{cases} 3(6s-4t) + 2(-4s+6t) = 3-10 \\ 2(6s-4t) + 3(-4s+6t) = 2-15 \end{cases}$

$\Leftrightarrow \begin{cases} 10s = -7 \\ 10t = -13 \end{cases} \Leftrightarrow \begin{cases} s = -\frac{7}{10} \\ t = -\frac{13}{10} \end{cases}$

What is the distance between the two lines?

$$||x(s) - y(t)|| = \left|\left| \begin{bmatrix} -\frac{17}{10} - \frac{23}{10} \\ -\frac{4}{10} + \frac{6}{10} \\ -\frac{7}{10} + \frac{3}{10} \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} -4 \\ \frac{2}{10} \\ -\frac{4}{10} \end{bmatrix} \right|\right| =$$

$$= \sqrt{16 + \frac{4}{100} + \frac{16}{100}} = \sqrt{16 + \frac{1}{5}} = \frac{\sqrt{81}}{\sqrt{5}} = \frac{9}{\sqrt{5}}$$

3. (20 points) Consider a multivariate function

$$f(x, y, z) = e^{x^2+y^3+z^4+4}.$$

(a) Find the gradient $\nabla f(x, y, z)$.

$$\nabla f = \begin{bmatrix} 2x f \\ 3y^2 f \\ 4z^3 f \end{bmatrix}$$

(b) Let $\boldsymbol{v} = \frac{1}{3}(1, 2, 2)^T$ and $\boldsymbol{x}_0 = (2, -2, 0)^T$. Find the directional derivative of f along $\boldsymbol{v}$ at $\boldsymbol{x}_0$?

$$\nabla f(x_0) = [4, 12, 0]^T$$

$$\nabla f(x_0) = \frac{1}{3}(4 + 24) = \underline{\underline{\frac{28}{3}}}$$

(c) Find the maximal rate of increase of f at x_0.

$$\|\nabla f\| = 4 \, \|[1, 3, 0]^T\| = 4\sqrt{10}$$

(d) Find the unit vector specifying the direction of the maximal decrease.

$$-\frac{1}{4\sqrt{10}} [4, 12, 0]^T = -\frac{1}{\sqrt{10}} [1, 3, 0]$$

4. (20 points) Consider the curve given by $x^2+y^2+3xy+1=0$. Find all points on the curve where the tangent line is parallel to $\boldsymbol{v}=(1,1)^T$.

Let $f(x,y) = x^2+y^2+3xy+1$

$f_x = 2x+3y$ 3'

$f_y = 2y+3x$ 3'

tangent $\parallel v \iff \nabla f \perp v \iff$

$$\begin{bmatrix} 2x+3y \\ 2y+3x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \iff 5x+5y=0$$

$$\iff y=-x$$

Then $x^2+x^2-3x^2+1=0 \iff x^2=1$

$\iff \left[\begin{array}{l} x=1 \Rightarrow y=-1 \\ x=-1 \Rightarrow y=1 \end{array}\right.$

<u>Answer</u>: $(1,-1)$ and $(-1,1)$

5. (20 points) Let $f(x,y) = x^3 - 6xy + y^3$. Find the gradient of f, the critical points, and Hessian at these points. For each point determine whether it is a local maximum, local minimum, or saddle point.

$f_x = 3x^2 - 6y$, $\quad f_y = 3y^2 - 6x$

$f_{xx} = 6x$, $\quad f_{xy} = -6$, $\quad f_{yy} = 6y$

Critical points: $\begin{cases} x^2 = 2y \\ y^2 = 2x \end{cases} \Rightarrow \begin{cases} y = \frac{1}{2}x^2 \\ \frac{1}{4}x^4 = 2x \end{cases}$

$x^4 = 8x \Rightarrow \left[\begin{array}{l} x = 0 \qquad\qquad\qquad \Rightarrow y = 0 \\ x^3 = 8 \Rightarrow x = 2 \Rightarrow y = 2 \end{array}\right.$

Critical points: $(0,0)$, $(2,2)$

$H_f = \begin{bmatrix} 6x & -6 \\ -6 & 6y \end{bmatrix}$

$H_f(0,0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$, $D = -36 \Rightarrow$ saddle point

$H_f(2,2) = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}$, $D = 144 - 36 > 0 \Rightarrow$ minimum

$f_{xx} > 0$

6. (extra credit)

Let $\boldsymbol{x}(s) : \mathbb{R} \to \mathbb{R}^2$ give an arc-length parametrization of a planar curve and suppose its curvature $\kappa(s) = 1$ for all $s \in \mathbb{R}$.

(a) Prove that this curve is a circle of radius 1.

Arclength parametrization $\Rightarrow \vec{T}(s) = \vec{x}'(s)$

$\kappa(s) = 1 \Leftrightarrow \|\vec{x}''(s)\| = 1$ since $\kappa(s) = \|\vec{T}'(s)\|$

$$\begin{cases} \|\vec{x}'(s)\| = 1 \\ \|\vec{x}''(s)\| = 1 \end{cases} \Leftrightarrow \begin{cases} (x'(s))^2 + (y'(s))^2 = 1 & (1) \\ (x''(s))^2 + (y''(s))^2 = 1 & (2) \end{cases}$$

Differentiate (1) $\Rightarrow 2x'(s)x''(s) = -2y'(s)y''(s) \Rightarrow$

$\Rightarrow (x')^2(x'')^2 = (y')^2(y'')^2 \xrightarrow{(1)\text{ and }(2)} (x')^2(x'')^2 = (1-(x')^2)(1-(x'')^2) \Rightarrow$

$\Rightarrow (x'')^2 + (x')^2 = 1 \Rightarrow \ldots$ Similarly, $(y'')^2 + (y')^2 = 1$

(b) Suppose $\boldsymbol{x}(0) = \boldsymbol{x}_0 = (0, 0)$ and the unit tangent $T(\boldsymbol{x}_0) = (0, 1)$. What point is the center of the circle? Explain.

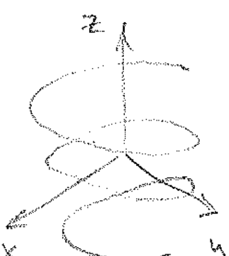
Provided initial conditions $\vec{x}(0) = (0,0)$ and $\vec{x}'(0) = (0,1)$ there are two cases $\vec{x}''(0) = (1,0)$ and $\vec{x}''(0) = (-1,0)$

Case 1: $\vec{x}''(0) = (-1,0)$ the solution to would be $\vec{x}(s) = [\cos s - 1, \sin s]^T$

Case 2: $\vec{x}''(0) = (1,0) \Rightarrow \vec{x}(s) = [1 - \cos s, \sin s]^T$

Case 1: center $= (-1,0)$, case 2: center $= (1,0)$.

(c) Is (a) true for a curve in the space $\mathbb{R}^3$? If so, then prove, else give an example (of a non-circle).



Let $\vec{x}(s) = \begin{bmatrix} \alpha \cos(\beta s) \\ \alpha \sin(\beta s) \\ \gamma s \end{bmatrix}$, (here: $\alpha > 0, \beta > 0, \gamma > 0$) $\quad \vec{x}'(s) = \begin{bmatrix} -\alpha\beta \sin(\beta s) \\ \alpha\beta \cos(\beta s) \\ \gamma \end{bmatrix}$

If we want $\|\vec{x}'(s)\| = 1 \Rightarrow \boxed{\alpha^2\beta^2 + \gamma^2 = 1}$

Now, $\kappa(s) = \|x''(s)\| = \|[-\alpha\beta^2 \cos(\beta s), -\alpha\beta^2 \sin(\beta s), 0]^T\|$

$= \alpha\beta^2$. If $\kappa(s) = 1$, $\boxed{\alpha\beta^2 = 1}$

Pick α, β, γ satisfying boxed equations: e.g. $\gamma = \frac{1}{2}$

$\Rightarrow \alpha = \frac{3}{4}$

$\Rightarrow \beta = \frac{2}{\sqrt{3}}$