Math 2605, **Exam#2**, Oct 28, 2010, **NAME:** Solutions

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points							
maximum	20	20	20	20	20	15	

Neither cheatsheets nor calculators are allowed.

1. Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function.
   (a) When is a point $\boldsymbol{x} \in \mathbb{R}^2$ called *critical*?

   when $\nabla f(\vec{x}) = 0$

   (b) When is a point $\boldsymbol{x}$ ~~is~~ called a local minimum?

   If $f(\vec{y}) \geq f(\vec{x})$ for all $\vec{y}$ in a nbhd of $\vec{x}$.

   (c) Does "critical" imply "local minimum or maximum"?

   No. (Think of a saddle point)

   Let $f(x, y) : D \to \mathbb{R}$ be a differentiable function on

   $$D = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| \leq 1\} \subset \mathbb{R}^2,$$

   the unit disk. Which statements are correct? (circle your answer)

   (a) The global maximum value is attained on the boundary of D. (True/false)
   (b) The global minimum can occur at a critical point of f. (True/false)
   (c) There could be more than one point in D where the global maximum is attained. (True/false)
   (d) There could be more than one point on the boundary of D where the global minimum is attained. (True/false)

2. Find the value of the global maximum/mimimum and the points where these are attained for

$$f(x,y) = x^2 + y^2$$

in the region D defined by

$$g(x,y) = x^4 + 7x^2y^2 + y^4 \le 1.$$

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ Critical point: $(0,0)$

$\nabla g = \begin{bmatrix} 4x^3 + 14xy^2 \\ 4y^3 + 14x^2y \end{bmatrix}$

$\det[\nabla f, \nabla g] = 0 \iff x(2y^3 + 7x^2y) - y(2x^3 + 7xy^2) = 0$

$\iff xy(2y^2 + 7x^2 - 2x^2 - 7y^2) = 5xy(x^2 - y^2)$

$= 5xy(x-y)(x+y) = 0$

4 cases: 1) $x = 0 \Rightarrow g(0,y) = 1 \Rightarrow y^4 = 1 \Rightarrow y = \pm 1$

2) $y = 0 \Rightarrow x^4 = 1 \Rightarrow x = \pm 1$

3) $\cancel{y} = x \Rightarrow 9x^4 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

4) $y = -x \Rightarrow$ $x = \pm \frac{1}{\sqrt{3}}$

Values at possible extrema:

$f(0,0) = 0, \quad f(0,1) = f(0,-1) = f(-1,0) = f(1,0) = 1$

$f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{2}{3}$

Answer: max. value $= 1$ attained at $(0,1), (1,0), (-1,0), (1,0)$

min. value $= 0$ attained at $(0,0)$.

3. (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 = 4 \iff \begin{cases} \lambda_1 = 3 \\ \lambda_2 = -1 \end{cases}$$

For $\lambda_1 = 3$, $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$, $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1$, $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{0}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) Write down an orthogonal matrix U such that $U^T AU$ is diagonal.

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(c) Perform the first step of Jacobi's diagonalization algorithm for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$G_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad G_1^T A G = G_1^T \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & \sqrt{2} & 1 \\ 2 & 4\sqrt{2} & 0 \\ 3 & \sqrt{2} & -1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{2})^2}\begin{bmatrix} 6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 8 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4. Let $A = VDU^T$ be a singular value decomposition of a 5×3 matrix A of rank 2.

(a) Define the generalized inverse A^+ of A.

$A^+ = UD^{-1}V^T$

(b) What is the size of the matrix A^+?

3×5

(c) How would you find a least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ via A^+?

$\vec{x} = A^+ b$

(d) What are the sizes of vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ above?

$\vec{x}$ is a 3-vector
$\vec{b}$ is a 5-vector

For the decomposition above, which statements are correct? (circle your answer)

(a) The rank of V is 2. (True/false)
(b) The rank of U is 3. (True/false)
(c) The norm on A equals the norm of D. (True/false)

5. Let

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 10 & 10 \\ 1 & -1 \end{bmatrix}.$$

(a) Find a singular value decomposition $A = VDU^T$.

$$A^TA = \frac{1}{2}\begin{bmatrix} 101 & 99 \\ 99 & 101 \end{bmatrix}$$

$$\begin{vmatrix} 101-2\mu & 99 \\ 99 & 101-2\mu \end{vmatrix} = (101-2\mu)^2 - 99^2 = 0$$

$$\mu_1 = 100, \quad \mu_2 = 1, \qquad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\underline{D = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}}, \quad \underline{U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}, \quad V = AUD^{-1} =$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 20 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(b) Find the rank 1 approximation $A_{(1)}$ of A.

$$A_{(1)} = 10 \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 10 & 10 \\ 0 & 0 \end{bmatrix}$$

(c) What does $\|A - A_{(1)}\|$ equal?

$\|$

1, the second singular value.

6. (**Bonus question**: 15 points) Consider the following system of equations that depends on a parameter $a \in \mathbb{R}$:

$$\begin{aligned} x^2 + ay &= 0 \\ y^2 + az &= 0 \\ z^2 - 2 &= 0 \end{aligned}$$

Execute one step of Newton's method starting with the initial guess $\boldsymbol{x}_0 = (a, a, a)^T$.

$$F = \begin{bmatrix} x^2 + ay \\ y^2 + az \\ z^2 - 2 \end{bmatrix}, \qquad J_F = \begin{bmatrix} 2x & a & 0 \\ 0 & 2y & a \\ 0 & 0 & 2z \end{bmatrix}$$

$$\boxed{\vec{x}_1 = \vec{x}_0 - J_F^{-1}(x_0) F(x_0)}$$

$$J_F(x_0) = \begin{bmatrix} 2a & a & 0 \\ 0 & 2a & a \\ 0 & 0 & 2a \end{bmatrix} = a \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \rightsquigarrow$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \Rightarrow J_F^{-1}(x_0) = \frac{1}{8a} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} a \\ a \\ a \end{bmatrix} - \frac{1}{8a} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2a^2 \\ 2a^2 \\ a^2 - 2 \end{bmatrix} =$$

$$= \begin{bmatrix} a - \frac{1}{8a}(5a^2 - 2) \\ a - \frac{1}{8a}(6a^2 + 4) \\ a - \frac{1}{8a}(4a^2 - 8) \end{bmatrix} = \begin{bmatrix} \frac{3}{8}a + \frac{1}{4a} \\ \frac{1}{4}a - \frac{1}{2a} \\ \frac{1}{2}a + \frac{1}{a} \end{bmatrix}$$