# Chapter 2 of Calculus$^{++}$

## The differential calculus in three or more variables

by

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## Section 1: Iterative solution of nonlinear systems of equations

### 1.1 Newton's method

Consider the following system of non linear equations:

$$\begin{aligned} x^2 + 2yx &= 4 \\ xy &= 1 \ . \end{aligned} \tag{1.1}$$

Any system of two equations in two variables can be written in the form

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \ . \end{aligned} \tag{1.2}$$

In this case we define $f(x, y) = x^2 + 2xy - 4$ and $g(x, y) = xy - 1$. All you have to do is to take whatever is on the right hand side of each equations, and subtract it off of both sides, leaving zero on the right. Just so we can standardize our methods, we shall always assume our equations have zero on the right hand side. If you run into one that doesn't, you know what to do as your first step: Cancel off the right hand sides.

Next, introducing $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix}$, we can write this as a single vector equation

$$\mathbf{F}(\mathbf{x}) = 0 \ . \tag{1.3}$$

In the case of (1.1), we have

$$\mathbf{F}(x, y) = \begin{bmatrix} x^2 + 2yx - 4 \\ xy - 1 \end{bmatrix} \ . \tag{1.4}$$

In this case, we can solve (1.3) by algebra alone. For this purpose, the original formulation is most convenient. Using the second equation in (1.1) to eliminate $y$, the first equation becomes $x^2 = 2$. Hence $x = \pm\sqrt{2}$. The second equation says that $y = 1/x$ and so we have two solutions

$$(\sqrt{2}, 1/\sqrt{2}) \qquad \text{and} \qquad (-\sqrt{2}, -1/\sqrt{2}) \ .$$

In general, it may be quite hard to eliminate either variable, and algebra alone cannot deliver solutions.

There is a way forward: Newton's method is a very effective algorithm for solving such equations. This is a "successive approximations method". It takes a starting guess for the solution $\mathbf{x}_0$, and iteratively improves the guess. The iteration scheme produces an *infinite sequence* of approximate solutions $\{\mathbf{x}_n\}$. Under favorable circumstances, this sequence will converge *very rapidly* toward an exact solution. In fact, the number of correct digits $x_n$ and $y_n$ will more or less double double at each step. If you have one digit right at the outset, you may expect about a million correct digits after 20 iterations – more than you are ever likely to want to keep!

To explain the use of Newton's method, we have to cover three points:

*(i)* How one picks the starting guess $\mathbf{x}_0$.

*(ii)* How the iterative loop runs; i.e., the rule for determining $\mathbf{x}_{n+1}$ given $\mathbf{x}_n$.

*(iii)* How to break out of the iterative loop – we need a "stopping rule" that ensures us our desired level of accuracy has been achieved when we stop iterating.

We begin by explaining *(ii)*, the nature of the loop. Once we are familiar with it, we can better understand what we have to do to start it and stop it.

The basis of the method is the linear approximation formula for $\mathbf{F}$ at $\mathbf{x}_0$:

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + J_{\mathbf{F}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \ . \tag{1.5}$$

Using this, we replace (1.3) with the approximate equation

$$\mathbf{F}(\mathbf{x}_0) + J_{\mathbf{F}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = 0 \ . \tag{1.6}$$

Don't let the notation obscure the simplicity of this: $\mathbf{F}(\mathbf{x}_0)$ is just a constant vector in $I\!R^2$ and $J_{\mathbf{F}}(\mathbf{x}_0)$ is just a constant $2 \times 2$ matrix. Using the shorter notation $\mathbf{F}(\mathbf{x}_0) = \mathbf{b}$ and $J_{\mathbf{F}}(\mathbf{x}_0) = A$, we can rewrite (1.6) as

$$A(\mathbf{x} - \mathbf{x}_0) = -\mathbf{b} \ .$$

We know what to do with this! We can solve this by row reduction. In fact, if $A$ is invertible, we have $\mathbf{x} - \mathbf{x}_0 = A^{-1}\mathbf{b}$, or, what is the same thing,

$$\mathbf{x} = \mathbf{x}_0 - A^{-1}\mathbf{b} \ .$$

Writing this out in the full notation, we have a formula for the solution of (1.6)

$$\mathbf{x} = \mathbf{x}_0 - \left(J_{\mathbf{F}}(\mathbf{x}_0)\right)^{-1} \mathbf{F}(\mathbf{x}_0) \ . \tag{1.7}$$

We now define $\mathbf{x}_1$ to be this solution. To get $\mathbf{x}_2$ from $\mathbf{x}_1$, we do the same thing starting from $\mathbf{x}_1$. In general, we define $\mathbf{x}_{n+1}$ to be the solution of

$$\mathbf{F}(\mathbf{x}_n) + J_{\mathbf{F}}(\mathbf{x}_n)(\mathbf{x} - \mathbf{x}_n) = 0 \ . \tag{1.8}$$

If $J_{\mathbf{F}}(\mathbf{x}_n)$ is invertible, this gives us

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \left(J_{\mathbf{F}}(\mathbf{x}_n)\right)^{-1} \mathbf{F}(\mathbf{x}_n) \ . \tag{1.9}$$

Now let's run through an example.

**Example 1 (Using Newton's iteration)** Consider the system of equations $\mathbf{F}(\mathbf{x}) = 0$ where $\mathbf{F}$ is given by (1.4). We will choose a starting point so that at least one of the equations in the system is satisfied, and the other is not *too* far off. This seems reasonable enough. Notice that with $x = y = 1$, $xy - 1 = 0$, while $x^2 - 2xy - 4 = -1$. Hence with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we have

$$\mathbf{F}(\mathbf{x}_0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} .$$

Now let's write our system in the form $F(x, y) = 0$. We can do this with

$$F(\mathbf{x}) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x^2 + 2yx - 4 \\ xy - 1 \end{bmatrix} .$$

Computing the Jacobian, we find that

$$J_F(\mathbf{x}) = \begin{bmatrix} 2x + 2y & 2x \\ y & x \end{bmatrix} , \tag{1.10}$$

and hence

$$J_F(\mathbf{x}_0) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} , \tag{1.11}$$

Hence (1.9) is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} .$$

Since

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} ,$$

we find

$$\mathbf{x}_1 = [\,3/2, 1/2\,] .$$

Notice that $\mathbf{x}_1$ is indeed considerably closer to the exact solution $\left[\sqrt{2}, 1/\sqrt{2}\right]$ than $\mathbf{x}_0$. Moreover,

$$F(\mathbf{x}_1) = -\frac{1}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

This is a better approximate solution; it is much closer to the actual solution. If you now iterate this further, you will find a sequence of approximate solutions converging to the exact solution $(\sqrt{2}, 1/\sqrt{2})$. You should compute $\mathbf{x}_2$ and $\mathbf{x}_3$ and observe the speed of convergence.

## 1.2 Choosing a starting point for Newton's method

With two variables, we can use what we know about generating plots of implicitly defined curves to locate good starting points. In fact, we can use such plots to determine the number of solutions. To do this, write $\mathbf{F}$ in the form

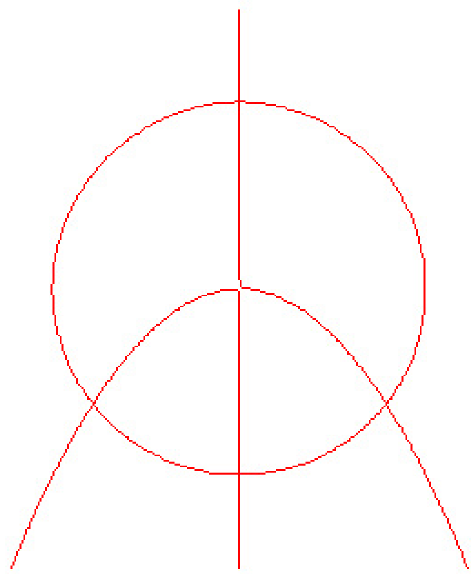
$$\mathbf{F}(x, y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} .$$

Each of the equations

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) = 0$$

is an implicit definition of a curve. Points where the two curves intersect are points belonging to the solution set of both equations; i.e., to the solution set of $\mathbf{F}(\mathbf{x}) = 0$.

**Example 2 (Using a graph to find a starting point for Newton's iteration)** Consider the system $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = 0$ where $f(x,y) = x^3 + xy$, and $g(x,y) = 1 - y^2 - x^2$. This is non linear, but simple enough that we can easily plot the curves. The equation $g(x,y) = 0$ is equivalent to $x^2 + y^2 = 1$, which is the equation for the unit circle. Since $f(x,y) = x(x^2 + y)$, $f(x,y) = 0$ is and only if $x = 0$, which is the equation of the $y$ axis, or $y = -x^2$, which is the equation of a parabola. Here is a graph showing the intersection of the implicitly defined curves:



The axes have been left off since one branch of the second curve is the $y$ axis. Since one curve is the unit circle though, you can easily estimate the coordinates of the intersections anyway. As you see, there are exactly 4 solutions. Two of them are clearly the exact solutions $(0, \pm 1)$. The other two are where the parabola crosses the circle. Carefully measuring on the graph, you could determine (axes would now help) that $y \approx -0.618$ and $x \approx \pm 0.786$. This would give us tow good approximate solutions. applying Newton's method, we could improve them to compute as many digits as we desire of the exact solution.

If you have more than two variables, graphs become harder to use. An alternative to drawing the graph is to evaluate $\mathbf{F}(\mathbf{x})$ at all of the points in some grid, in some limited range of the variables. Use whichever grid points give $\mathbf{F}(\mathbf{x}) \approx= 0$ as your starting points.

### 1.3 Geometric interpretation of Newton's method

Newton's method is based on the tangent plane approximation, and so it has a geometric interpretation. This will help us to understand why it works when it does, and how we can reliably stop it.

Here is how this goes for the system

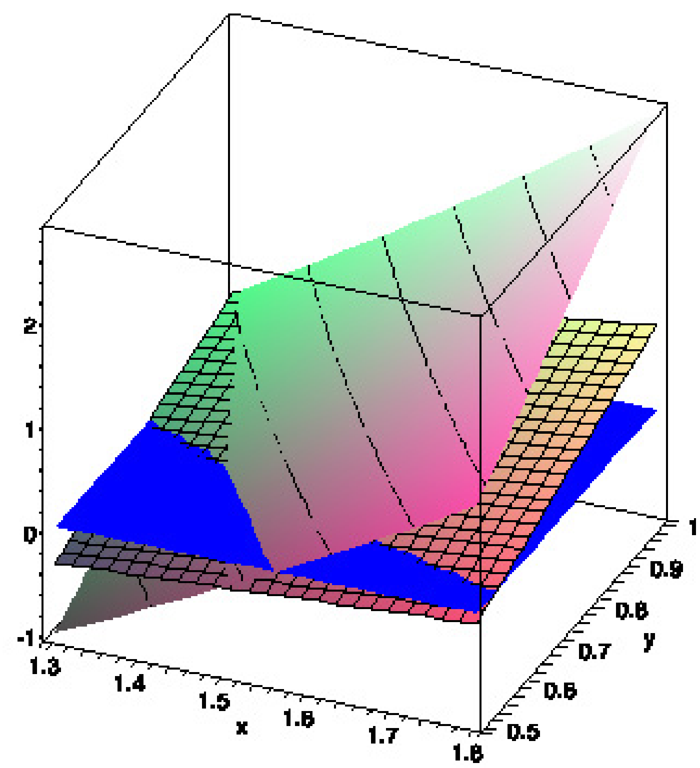
$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \ . \end{aligned} \tag{1.12}$$

Replace this by the equivalent system

$$
\begin{aligned}
z &= f(x,y) \\
z &= g(x,y) \\
z &= 0 \ .
\end{aligned}
\tag{1.13}
$$

From an algebraic standpoint, we have taken a step backwards – we have gone from two equations in two variables to three equations in three variables. However, (1.13) has an interesting geometric meaning: The graph of $z = f(x,y)$ is a surface in $I\!R^3$, as is the graph of $z = g(x,y)$. The graph of $z = 0$ is just the $x, y$ plane – a third surface. Hence the solution set of (1.13) is given by the intersection of 3 surfaces.

For example, here you a plot of the three surfaces in (1.13) when $f(x,y) = x^2 + 2xy - 4$ and $g(x,y) = xy - 1$, as in Example 1. Here, we have plotted $1.3 \le x \le 1.8$ and $0.5 \le y \le 1$, which includes one exact solution of the system (1.12) in this case. The plane $z = 0$ is the surface in solid color, $z = f(x,y)$ shows the contour lines, and $z = g(x,y)$ is the surface showing a grid. You see where all three surfaces intersect, and that is the where the solution lies.




You also see in this graph that the tangent plane approximation is pretty good in this region, so replacing the surfaces by their tangent planes will not wreak havoc on the graph. So here is what we do: Take any point $(x_0, y_0)$ so that the three surface intersect *near* $(x_0, y_0, 0)$. Then replace the surfaces $z = f(x,y)$ and $z = g(x,y)$ by their tangent planes at $(x_0, y_0)$, and compute the intersection of the tangent planes with the plane $z = 0$. This is a linear algebra problem, and hence is easily solved. Replacing $z = f(x,y)$ and $z = g(x,y)$

by the equations of their tangent planes at $(x_0, y_0)$ amounts to the replacement

$$z = f(x,y) \qquad \to \qquad z = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$$

and

$$z = g(x,y) \qquad \to \qquad z = g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$$

where $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. This transforms (1.13) into

$$\begin{aligned} z &= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) \\ z &= g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) \\ z &= 0\ . \end{aligned} \tag{1.14}$$

Now we can eliminate $z$, and pass to the simplified system

$$\begin{aligned} f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) &= 0 \\ g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) &= 0\ . \end{aligned} \tag{1.15}$$

Since $J_{\mathbf{F}}(\mathbf{x}_0) = \begin{bmatrix} \nabla f(\mathbf{x}_0) \\ \nabla g(\mathbf{x}_0) \end{bmatrix}$, this is equivalent to (1.6) by the usual rules for matrix multiplication.

We see from this analysis that how close we come to an exact solution in one step of Newton's method depends on, among other things, how good the tangent plane approximation is at the current approximate solution. We know that tangent plane approximations are good when the norm of the Hessian is not too large. We can also see that there will be trouble if $J_{\mathbf{F}}$ is not invertible, or even if it $\nabla f$ and $\nabla g$ are nearly proportional, in which case $(J_{\mathbf{F}})^{-1}$ will have a large norm. There is a precise theorem, due the the 20th century Russian mathematician Kantorovich that can be paraphrased as saying that if $\mathbf{x}_0$ is not too far from an exact solution, $\|(J_{\mathbf{F}})^{-1}\|$ is not too large, and each component of $\mathbf{F}$ has a Hessian that is not too large, Newton's method works and converges very fast. The precise statement makes it clear what "not too large" means. We will neither sate it nor prove it here – it is quite intricate, and in practice one simply uses the method as described above, and stops the iteration when the answers stop changing in the digits that one is concerned with.

**1.4 More variables**

We have explained everything so far in the case of two equations in two variables. But is $\mathbf{F}$ is a functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, so that $\mathbf{F}(\mathbf{x}) = 0$ is a system of $m$ equations in $n$ variables, the passage to the approximate linear system

$$\mathbf{F}(\mathbf{x}_0) + J_{\mathbf{F}}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) = 0\ . \tag{1.16}$$

by way of the linear approximation

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + J_{\mathbf{F}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \ . \tag{1.17}$$

is just as valid. We may therefore recursively define the sequence $\{\mathbf{x}_n\}$ by

$$J_{\mathbf{F}}(\mathbf{x}_0)(\mathbf{x}_{n+1} - \mathbf{x}_n) = -\mathbf{F}(\mathbf{x}_0) \ . \tag{1.18}$$

Notice we do not write this in terms of the inverse of $J_{\mathbf{F}}$ any longer – indeed, when $m \neq n$, the Jacobian will not be square, and there will be no inverse. As long as there are more variables than equations though, we can hope that the system in (1.18) is underdetermined, and hence solvable. We can proceed as before.

**Problems**

**1** Let $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix}$ where $f(x,y) = x^3 + xy$, and $g(x,y) = 1 - 4y^2 - x^2$. Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

**a** Compute $J_{\mathbf{F}}(\mathbf{x})$ and $J_{\mathbf{F}}(\mathbf{x}_0)$.

**b** Use $\mathbf{x}_0$ as a starting point for Newton's method, and compute the next approximate solution $\mathbf{x}_1$.

**c** Evaluate $\mathbf{F}(\mathbf{x}_1)$, and compare this with $\mathbf{F}(\mathbf{x}_0)$.

**d** Draw graphs of the curves implicitly defined by $f(x,y) = 0$ and $g(x,y) = 0$. How many solutions are there of this non linear system?

**2** Let $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix}$ where $f(x,y) = \sqrt{x} + \sqrt{y} - 3$, and $g(x,y) = x^2 + 4y^2 = 18$.

**a** Compute $\mathbf{F}(\mathbf{x}_0)$ for $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. does this look like a reasonable starting point? Compute $J_{\mathbf{F}}(\mathbf{x}_0)$. What happens if you try to use $\mathbf{x}_0$ as your starting point for Newton's method?

**b** Draw graphs of the curves implicitly defined by $f(x,y) = 0$ and $g(x,y) = 0$. How many solutions are there of this non linear system? Find starting points $\mathbf{x}_0$ near each of them with integer entries.

**c** Let $\mathbf{x}_0$ be the starting point that you found in part (b) that is closest to the $x$-axis. Compute the next approximate solution $\mathbf{x}_1$.

**d** Evaluate $\mathbf{F}(\mathbf{x}_1)$, and compare this with $\mathbf{F}(\mathbf{x}_0)$.

**3** Let $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix}$ where $f(x,y) = \sin(xy) - x$, and $g(x,y) = x^2y - 1$. Let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**a** Compute $J_{\mathbf{F}}(\mathbf{x})$ and $J_{\mathbf{F}}(\mathbf{x}_0)$.

**b** Use $\mathbf{x}_0$ as a starting point for Newton's method, and compute the next approximate solution $\mathbf{x}_1$.

**c** Evaluate $\mathbf{F}(\mathbf{x}_1)$, and compare this with $\mathbf{F}(\mathbf{x}_0)$.

**d** How many solutions of this system are there in the region $-2 \leq x \leq 2$ and $0 \leq y \leq 10$? Compute each of them to 10 decimal places of accuracy – using a computer, of course.

## Section 2: Optimization problems

### 2.1 What is an optimization problem?

A *optimization problem* in $n$ variables is one in which we are given a function $f(\mathbf{x})$, and a set $D$ in $\mathbb{R}^n$ of *admissible points*, and we are asked to find either the maximum or minimum value of $f(\mathbf{x})$ as $\mathbf{x}$ ranges over $D$.

If $D$ is a bounded and closed subset, and if $f$ is continuous, then there is always point $\mathbf{x}_1$ in $D$ with the property that

$$f(\mathbf{x}_1) \geq f(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \text{ in } D\ , \tag{2.1}$$

and there is always a point $\mathbf{x}_0$ in $D$ with the property that

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \text{ in } D\ . \tag{2.2}$$

The knowledge that such points exist, under these conditions, will play a basic role in our reasoning.

**Definition (Maximizer and minimizer)** Any point $\mathbf{x}_1$ satisfying (2.1) is called a *maximizer of $f$ in $D$*, and any point $\mathbf{x}_0$ satisfying (2.1) is called a *minimizer of $f$ in $D$*. The value of $f$ at a maximizer is the *maximum value of $f$ in $D$*, and the value of $f$ at a minimizer is the *minimum value of $f$ in $D$*.

The regions $D$ that we will be concerned with will be given by $m$ *constraints*, which are inequalities of the form

$$\begin{aligned} g_1(\mathbf{x}) &\leq 0 \\ g_1(\mathbf{x}) &\leq 0 \\ \vdots\ &\ \vdots \\ g_m(\mathbf{x}) &\leq 0 \end{aligned} \tag{2.3}$$

A point $\mathbf{x}$ belongs to if and only if $D$ if it satisfies *each* of these inequalities.

Many sets can be expressed in this way. We will now give a series of examples in three variables.

**Example 1 (Constraint inequalities for the unit ball and sphere in $\mathbb{R}^3$)** Let

$$g_1(x, y, z) = x^2 + y^2 + z^2 - 1\ .$$

The the closed unit ball in $\mathbb{R}^3$ the set of points satisfying

$$g_1(\mathbf{x}) \leq 0\ .$$

Next, let $D$ be the unit sphere in $\mathbb{R}^3$; i.e., the set of all unit vectors in $\mathbb{R}^3$. This is given by

$$g_1(\mathbf{x}) = 0\ . \tag{2.4}$$

While this does not seem to fit the pattern in (2.3), because it is an equation, and not a system of inequalities, we can take care of that. Let

$$g_2(\mathbf{x}) = -g_1(\mathbf{x}) \ . \tag{2.5}$$

Then

$$g_1(\mathbf{x}) = 0 \iff g_1(\mathbf{x}) \le 0 \quad \text{and} \quad g_2(\mathbf{x}) \le 0 \ .$$

Thus, (2.4) is equivalent to the system of inequalities

$$\begin{array}{c} g_1(\mathbf{x}) \le 0 \\ g_2(\mathbf{x}) \le 0 \end{array} \ . \tag{2.6}$$

As you can see from Example 1, we can include equality constraints in the framework of (2.3). It is a pretty general framework for the specification of closed sets in $I\!R^n$. Here is another example.

**Example 2 (More constraint specifications)** Let $D$ be the closed set of points in $I\!R^3$ that lies above the cone $z = |\mathbf{x}|/2$ and inside the unit ball. Then $D$ is given by

$$\begin{array}{c} g_1(\mathbf{x}) \le 0 \\ g_2(\mathbf{x}) \le 0 \end{array} \ . \tag{2.7}$$

where $g_1(\mathbf{x}) = |\mathbf{x}|^2 - 4z^2$ and $g_2(\mathbf{x}) = |\mathbf{x}|^2 - 1$. note that we could have used $|\mathbf{x}| - 2z$ for $g_1(\mathbf{x})$, but it is always easier to compute with squares of lengths of vectors, as we shall see.

When a set $D$ is specified as in (2.3), and each of the functions $g$ is continuous, the *interior* of $D$ is the set of all $\mathbf{x}$ for which

$$\begin{array}{c} g_1(\mathbf{x}) < 0 \\ g_1(\mathbf{x}) < 0 \\ \vdots \ \vdots \\ g_m(\mathbf{x}) < 0 \end{array} \ . \tag{2.8}$$

Notice that the difference with (2.3) lies in the fact that here all of the inequalities are *strict*. By the continuity of the functions $g_j$, if $\mathbf{x}$ is in the interior of $D$, then so is some open neighborhood of points around $\mathbf{x}$. (Of course, the interior can be the empty set, and it is when $D$ is the unit sphere.)

The *boundary* of $D$ is the set of all points $\mathbf{x}$ satisfying the system of equations

$$\begin{array}{c} g_1(\mathbf{x}) = 0 \\ g_1(\mathbf{x}) = 0 \\ \vdots \ \vdots \\ g_m(\mathbf{x}) = 0 \end{array} \ . \tag{2.9}$$

To solve an optimization problem is to find all maximizers and minimizers, if any, and the corresponding maximum and minimum values. Our goal in this section is to explain a strategy for doing this. We shall deal separately with the interior and the boundary.

The interior points are easy: If the gradient is nonzero, there is an "uphill direction" and a downhill direction" moving away from $\mathbf{x}_0$, and staying in $D$. This is incompatible with $\mathbf{x}_0$ being either a maximizer or a minimizer.

### 2.1 A strategy for solving optimization problems

**Theorem 1 (Critical points and interior optimizers)** *Let $D$ region in $I\!R^n$, and suppose that $\mathbf{x}_0$ is in the interior of $D$, and is either a minimizer or a maximizer. Then $\mathbf{x}_0$ is a critical point of $f$.*

The validity of this is probably quite clear. It is nonetheless worth going through a proof that dots the $i$'s and crosses the $t$'s. Doing so will help you appreciate the content of Theorem 2 from the previous section.

**Proof:** Let $f$ be a function defined on $I\!R^n$ with continuous first order partial derivatives. Suppose that $\mathbf{x}_0$ is in the interior of $D$, and $\nabla f(\mathbf{x}_0) \neq 0$. Let $\mathbf{v} = \nabla f(\mathbf{x}_0)$. Then the linear approximation to $f$ at $\mathbf{x}_0$ is

$$h(\mathbf{x}) = f(\mathbf{x}_0) + \mathbf{v} \cdot (\mathbf{x} - \mathbf{x}_0) \ . \tag{2.10}$$

Since $\mathbf{x}_0$ is in the interior, we have some "wriggle room", and there is an $r > 0$ so that

$$|t| < r \ \Rightarrow \ \mathbf{x}_0 + t\mathbf{v} \quad \text{is in} \quad D \ .$$

Now, by adding and subtracting $h$,

$$f(\mathbf{x}_0 + t\mathbf{v}) = h(\mathbf{x}_0 + t\mathbf{v}) + [f(\mathbf{x}_0 + t\mathbf{v}) - h(\mathbf{x}_0 + t\mathbf{v})] \ . \tag{2.11}$$

Then by (2.10),

$$h(\mathbf{x}_0 + t\mathbf{v}) = f(\mathbf{x}_0) + t|\mathbf{v}|^2 \ . \tag{2.12}$$

If $\mathbf{v} \neq 0$, then Theorem 2 of the previous section says that

$$\lim_{t \to 0} \frac{|f(\mathbf{x}_0 + t\mathbf{v}) - h(\mathbf{x}_0 + t\mathbf{v})|}{|t||\mathbf{v}|} = 0 \ .$$

In particular, there is a $t_0 > 0$ so that

$$|t| < t_0 \ \Rightarrow \ \frac{|f(\mathbf{x}_0 + t\mathbf{v}) - h(\mathbf{x}_0 + t\mathbf{v})|}{|t||\mathbf{v}|} \leq \frac{1}{2}|\mathbf{v}| \ ,$$

which is the same as

$$|t| < t_0 \ \Rightarrow \ |f(\mathbf{x}_0 + t\mathbf{v}) - h(\mathbf{x}_0 + t\mathbf{v})| \leq \frac{|t|}{2}|\mathbf{v}|^2 \ . \tag{2.13}$$

Now combining (2.11), (2.12) and (2.13), we see that

$$0 < t < t_0 \ \Rightarrow\ f(\mathbf{x}_0 + t\mathbf{v}) \geq f(\mathbf{x}_0) + \frac{t}{2}|\mathbf{v}|^2 \tag{2.14}$$

and

$$-t_0 < t < 0 \ \Rightarrow\ f(\mathbf{x}_0 + t\mathbf{v}) \leq f(\mathbf{x}_0) - \frac{t}{2}|\mathbf{v}|^2 \tag{2.15}$$

Therefore, for small positive values of $t$, $f(\mathbf{x}_0+t\mathbf{v}) > f(\mathbf{x}_0)$, so $\mathbf{x}_0$ cannot be a maximizer. Also, for small negative values of $t$, $f(\mathbf{x}_0+t\mathbf{v}) < f(\mathbf{x}_0)$, so $\mathbf{x}_0$ cannot be a minimizer. Hence condition $\nabla f(\mathbf{x}_0) \neq 0$ is incompatible with $\mathbf{x}_0$ being an optimizer. ■

Now we come to the boundary points. If $\mathbf{x}_0$ is on the boundary of $D$, we cannot move away from $\mathbf{x}_0$ in an arbitrary direction $\mathbf{v}$, and still stay on the boundary: We are only "allowed" to move in directions that do not change the values of any of the constraint functions $g_j$.

Here is the idea: Let $\mathbf{x}(t)$ be some parameterized path on the boundary of $D$ that passes through $\mathbf{x}_0$ at $t = 0$. Suppose that $\mathbf{x}'(0) = \mathbf{v}$. Then, by the chain rule, for each $j$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_j(\mathbf{x}(t))\bigg|_{t=0} = \mathbf{v} \cdot \nabla g_j(\mathbf{x}_0)\ .$$

But since the curve $\mathbf{x}(t)$ lies in the boundary of $D$, on which each $g_j$ is constant,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_j(\mathbf{x}(t))\bigg|_{t=0} = 0\ .$$

That is,

$$\mathbf{v} \cdot \nabla g_j(\mathbf{x}_0) = 0\ . \tag{2.16}$$

Direction vectors $\mathbf{v}$ satisfying (2.16) are called *allowed direction vectors.*

Now, if $\mathbf{x}_0$ is a maximizer or a minimizer for $f$ on the boundary of $D$, then $t = 0$ is a maximizer or a minimizer for the single variable function $f(\mathbf{x}(t))$. Hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=0} = \mathbf{v} \cdot \nabla f(\mathbf{x}_0)\ .$$

The key fact on which this argument depends is this:

• *Whenever* $\mathbf{v}$ *is an allowed direction at* $\mathbf{x}_0$*, there is a path* $\mathbf{x}(t)$ *lying in the boundary of* $D$ *such that* $\mathbf{x}(0) = \mathbf{x}_0$ *and such that* $\mathbf{x}'(0) = \mathbf{v}$

Previously, in two variables, we proved this using the Implicit Function Theorem. The same ideas will again lead to the same conclusion. Let us accept this for the moment.

We therefore conclude that whenever $\mathbf{x}_0$ is an optimizer for $f$ on the boundary of $D$,

$$\left\{ \mathbf{v}\cdot\nabla g_j(\mathbf{x}_0) = 0 \quad \text{for all } j \right\} \Rightarrow \mathbf{v}\cdot\nabla f(\mathbf{x}_0) = 0 \ . \tag{2.17}$$

**Lemma** *The statement (2.17) is equivalent to the statement that $\nabla f(\mathbf{x}_0)$ is a linear combination of the vectors $\nabla g_j(\mathbf{x}_0)$; i.e., that there exist numbers $\lambda_1,\ldots,\lambda_m$ so that*

$$\nabla f(\mathbf{x}_0) = \lambda_1\nabla g_1(\mathbf{x}_0) + \lambda_2\nabla g_2(\mathbf{x}_0) + \cdots + \lambda_m\nabla g_m(\mathbf{x}_0) \ . \tag{2.18}$$

**Proof:** Let $S$ be the subspace of $I\!R^n$ spanned by $\{\nabla g_1(\mathbf{x}_0),\ldots,\nabla g_m(\mathbf{x}_0)\}$, and let $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ be an orthonormal basis of $S^\perp$. Then

$$\{\nabla g_1(\mathbf{x}_0),\ldots,\nabla g_m(\mathbf{x}_0)\} \cup \{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$$

spans $I\!R^n$. In particular, there exist numbers $\lambda_1,\ldots,\lambda_m$ and $\mu_1,\ldots,\mu_k$ so that

$$\nabla f(\mathbf{x}_0) = \sum_{i=1}^m \lambda_i\nabla g_i(\mathbf{x}_0) + \sum_{j=1}^k \mu_j\mathbf{u}_j \ . \tag{2.19}$$

Now, since $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthonormal basis of $S^\perp$, and since each $\nabla g_i(\mathbf{x}_0)$ lies in $S$, for each $i$ and $j$

$$\mathbf{u}_j\cdot\nabla g_i(\mathbf{x}_0) = 0 \ . \tag{2.20}$$

In particular, each of the $\mathbf{u}_j$ are allowed directions. Therefore, $\mathbf{u}_j\cdot\nabla f(\mathbf{x}_0) = 0$ for each $j$. But by (2.19), (2.20) and the fact that $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthonormal set of vectors,

$$\mathbf{u}_j\cdot\nabla f(\mathbf{x}_0) = \mu_j \ .$$

Therefore $\mu_j = 0$ for each $j$, and thus (2.19) reduces to (2.18) ■

Now we have the following result:

**Theorem 2 (Lagrange multipliers and boundary optimizers)** *Let $D$ region in $I\!R^n$ and suppose that $\mathbf{x}_0$ is on the boundary of $D$, and is either a minimizers or a maximizer. Suppose that the boundary of $D$ is specified by a system of equation of the form (2.9). Then there exist numbers $\lambda_1,\ldots,\lambda_m$ so that*

$$\nabla f(\mathbf{x}_0) = \sum_{i=1}^m \lambda_i\nabla g_i(\mathbf{x}_0). \tag{2.21}$$

Notice that the vector equation (2.21) is a system of $n$ equations in $n+m$ unknowns, namely $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_m$. However, we also have the $m$ constraint equations (2.9), so altogether, we have $n+m$ equations for $n+m$ unknowns. Solving this system provides us with all possible "suspects" for maximizers and minimizers on the boundary.

We note that usually the set $D$ defined by (2.9) is only interesting when $m < n$. Indeed, if $m = n$, then (2.9) provides $n$ equations for $n$ variables, the $x_1, \ldots, x_n$, and so in this case, $D$ likely to consist of a finite set of points. One takes this whole set $D$ as the "list of suspects", and plugs them into $f$ to see which ones give the minimum and maximum values. If $m > n$, the system (2.9) is likely to be overdetermined, and $D$ may then be empty.

In particular, when $n = 2$, we are only interested in the case of one constraint equation $g(\mathbf{x}) = 0$, and then (2.21) reduces to

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$$

for some $\lambda$. But in this case, the columns of $[\nabla f(\mathbf{x}), \nabla g(\mathbf{x})]$ are linearly dependent, and we can check for this by solving

$$\det([\nabla f(\mathbf{x}), \nabla g(\mathbf{x})]) = 0 \ ,$$

as we usually did in the previous chapter.

We now give a large number of examples in three and four variables. Our first example will show how minimization problems in 4 or more variables can easily arise when considering geometric questions in $I\!R^2$ or $I\!R^3$. The example concerns the problem of finding the distance between two curves in the plane. Suppose that these are given implicitly by $g_1(\mathbf{x}) = 0$ and $g_2(\mathbf{x}) = 0$.

• *We seek to find points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *satisfying* $g_1(\mathbf{x}_1) = 0$ *and* $g_2(\mathbf{x}_2) = 0$ *with* $|\mathbf{x}_1 - \mathbf{x}_2|$ *as small as possible.*

Here is a natural way to solve this problem: Let $(x, y)$ and $(u, v)$ denote the coordinates of $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. Define

$$f(x, y, u, v) = (x-u)^2 + (y-v)^2$$

so that $f$ is the square of the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$. Our problem now is to minimize $f$, subject to the constraints

$$g_1(x, y) = 0$$

$$g_2(u, v) = 0$$

Thus, the problem fits exactly into the framework that we have been discussing in this section.

Now, if one of the curves is a line, there is an alternate approach leading to a problem with two variables and one constraint: A type of problem that we learned to solve in the previous chapter. The reason is that we have a formula for the distance form a point $(x, y)$ to a line. We can then minimize this distance over the second curve.

For example, suppose the line is given by $x + 2y = 4$. Then $\mathbf{x}_0 = [\,0, 2\,]$ is a point on the line , and the normal vector to the line is $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then the distance from $\mathbf{x}$ to the line is

$$\frac{1}{\sqrt{5}} \left| \begin{bmatrix} x \\ y-2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right| = \frac{|x+2-4|}{\sqrt{5}} \ .$$

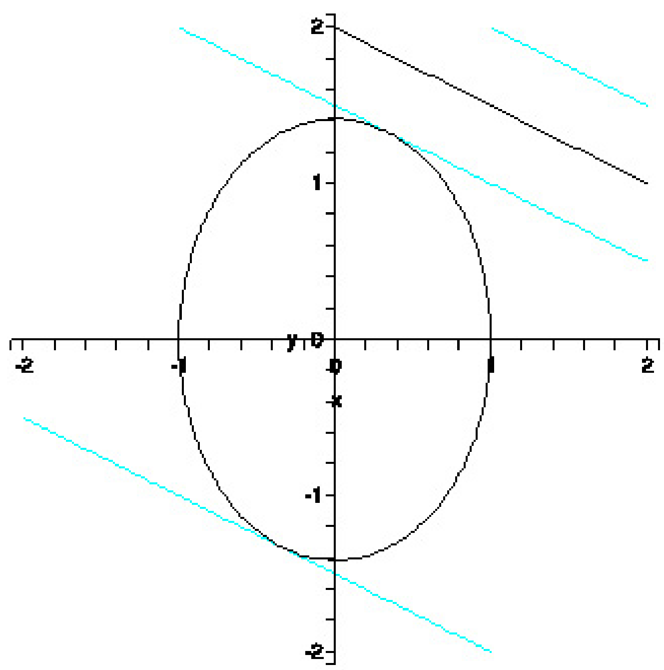
If the second curve is the ellipse $2x^2 + y^2 = 2$, we can then minimize

$$f(x, y) = (x + 2y - 4)^2/5$$

subject to the constraint

$$g(x, y) = 0 \qquad \text{where} \qquad g(x, y) = 2x^2 + y^2 - 2 \ .$$

Here is a graph showing the ellipse and and the line in black, with a few level curves of the distances function drawn in more lightly. These are of course lines parallel to the original line, and we have drawn than in for the levels that make them tangent to the ellipse. Evidently, the point of tangency in the upper right of the graph is the points on the ellipse that is closest to the line.



We now explain how to solve this problem without using the formula for the distance between a point and a line. This is almost as simple, and the approach also works when we replace the line by something more complicated, as we shall see in the next example. First, let us start with something relatively simple.

**Example 3 (The distance between an ellipse and a line)** We have already seen how to compute the distance between a point and a curve, but what about the distance between two curves?

To begin with a simple case, let us consider a line and an ellipse. Let the line be given by

$$x + 2y = 4$$

and the ellipse be given by

$$2x^2 + y^2 = 2 \ .$$

Let $(x, y)$ denote any point on the line, and let $(u, v)$ denote any point on the ellipse. We seek to minimize

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2 \ ,$$

which is the square of the distance between $(x, y)$ and $(u, v)$ as $(x, y)$ ranges over the line, and $(u, v)$ ranges over the ellipse.

If we define

$$g_1(x, y, u, v) = x + 2y = 4 \qquad \text{and} \qquad g_2(x, y, u, v) = 2u^2 + v^2 - 2 \ ,$$

then the system

$$\begin{aligned} g_1(x, y, u, v) &= 0 \\ g_2(x, y, u, v) &= 0 \end{aligned}$$

expresses the constraint that $(x, y)$ is to lie on the line, and $(u, v)$ on the ellipse.

It is geometrically clear that a minimum exists, but no maximum – there are points on the line that are arbitrarily far away from the ellipse.

To apply Theorem 2, we now compute

$$\nabla f = 2 \begin{bmatrix} x - u \\ y - v \\ u - x \\ v - y \end{bmatrix} \qquad \nabla g_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \nabla g_2 = \begin{bmatrix} 0 \\ 0 \\ 4u \\ 2v \end{bmatrix} \ .$$

Then according to Theorem 2, there exist numbers $\lambda_1$ and $\lambda_2$ such that

$$\begin{bmatrix} x - u \\ y - v \\ u - x \\ v - y \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 4u \\ 2v \end{bmatrix} \ . \tag{2.22}$$

Considering the first two entries, and then the last two, we see that

$$\begin{bmatrix} x - u \\ y - v \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x - u \\ y - v \end{bmatrix} = -\lambda_2 \begin{bmatrix} 4u \\ 2v \end{bmatrix} \ . \tag{2.23}$$

Notice that the vector $\begin{bmatrix} x - u \\ y - v \end{bmatrix}$ is not zero when $(x, y)$ is on the line, and $(u, v)$ is on the ellipse, since the line and the ellipse do not intersect. Therefore, neither $\lambda_1$ nor $\lambda_2$ is zero. Hence, the three vectors

$$\begin{bmatrix} x - u \\ y - v \end{bmatrix} \ , \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 4u \\ 2v \end{bmatrix} \ . \tag{2.24}$$

are all non zero and proportional.

From this we conclude that

$$\frac{x - u}{y - v} = \frac{1}{2} = \frac{2u}{v} \ . \tag{2.25}$$

The second equation in (2.25) says that

$$v = 4u \ , \tag{2.26}$$

Substituting this into the constraint equation $g_2 = 0$, we get $18u^2 = 2$. The is means the $u = \pm 1/3$, and then from (2.26), we find the two solutions

$$(u, v) = (1/3, 4/3) \qquad \text{and} \qquad (u, v) = (-1/3, -4/3)\ . \tag{2.27}$$

The first equation in (2.25) can be written as $2(x - u) = y - v)$ ,or $2x - y = 2u - v$. Combining this with the constraint equation $g_1 = 0$, we have the system

$$\begin{aligned} 2x - y &= 2u - v \\ x + 2y &= 4\ . \end{aligned} \tag{2.28}$$

For $(u, v) = (1/3, 4/3)$, (2.28) becomes

$$\begin{aligned} 2x - y &= -2/3 \\ x + 2y &= 4\ . \end{aligned}$$

One easily solves this system for $x$ and $y$, finding $x = 8/15$ and $y = 26/15$.

For $(u, v) = (-1/3, -4/3)$, (2.28) becomes

$$\begin{aligned} 2x - y &= 2/3 \\ x + 2y &= 4\ . \end{aligned}$$

One easily solves this system for $x$ and $y$, finding $x = 16/15$ and $y = 22/15$.

Thus, we have found two solutions of our system of equations

$$\frac{1}{15}\begin{bmatrix} 8 \\ 26 \\ 5 \\ 20 \end{bmatrix} \qquad \text{and} \qquad \frac{1}{15}\begin{bmatrix} 16 \\ 22 \\ -5 \\ -20 \end{bmatrix}\ .$$

Evaluating $f$ at these two points, we get $1/5$ and $49/5$ respectively. Evidently the distance between the line and the ellipse is $1/\sqrt{5}$.

To check our work graphically, note that our computation say that the points of tangency in the graph we drew right before this example should be given by (2.27). As you can see, this is what is shown in the graph.

There is an important general conclusion that can be drawn from our analysis in Example 3: Suppose that we have two curves given implicitly by $g_1 = 0$ and $g_2 = 0$. Suppose that the two curves do not intersect. (If they do, the distance between them is zero). Then that analysis leading from (2.22) to (2.24) leads to the conclusion that the three vectors

$$\begin{bmatrix} x - u \\ y - v \end{bmatrix} \quad , \quad \nabla g_1(x, y) \qquad \text{and} \qquad \nabla g_2(u, v)\ . \tag{2.29}$$

are all non zero and proportional.

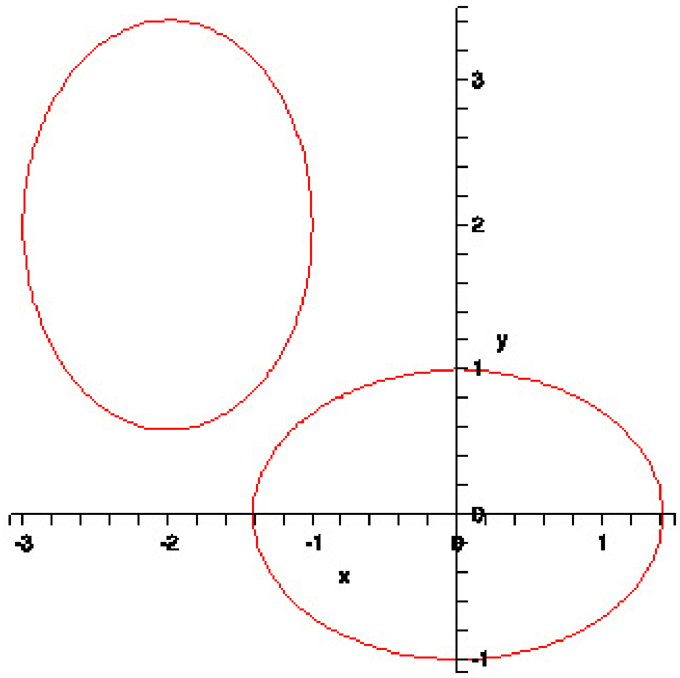
From this, we can extract a pair of equations, which, together with the two constraint equations, $g_1 = 0$ and $g_2 = 0$, give us four equations in four unknowns. Solving for these gives us our "list of suspects".

In the next example, we apply this find the distance between two ellipses.

**Example 4 (The distance between two ellipses)** Let $g_1$ and $g_2$ be given by

$$g_1(x,y) = 2(x+2)^2 + (y-2)^2 - 2 \qquad \text{and} \qquad g_2(u,v) = u^2 + 2v^2 - 2 \ .$$

Here is a graph showing these two ellipses, which do not intersect:




We next compute

$$\nabla g_1(x,y) = \begin{bmatrix} 4x+8 \\ 2y-4 \end{bmatrix} \qquad \text{and} \qquad \nabla g_2(u,v) = \begin{bmatrix} 2u \\ 4v \end{bmatrix} \ .$$

Hence from (2.29) we conclude that

$$\det\left(\begin{bmatrix} x-u & 2x-4 \\ y-v & y-2 \end{bmatrix}\right) = 0 \qquad \text{and} \qquad \det\left(\begin{bmatrix} x-u & 2u \\ y-v & v \end{bmatrix}\right) = 0 \ .$$

That is,

$$(x-u)(y-2) - (2x-4)(y-v) = 0 \qquad \text{and} \qquad (x-u)v - (y-v)2u = 0 \ .$$

Together with the constraint equations, we now have the system

$$\begin{aligned} (x-u)(y-2) - (2x-4)(y-v) &= 0 \\ (x-u)v - (y-v)2u &= 0 \\ 2(x+2)^2 + (y-2)^2 - 2 &= 0 \\ u^2 + 2v^2 &= 0 \end{aligned}$$

This is a system of quadratic equations, and so it can be solved explicitly. A more efficient way to learn the answer, to as many decimal places as we desire, is to apply Newton's method.

Let $\mathbf{X}$ denote the vector in $I\!R^4$ given by $\mathbf{X} = \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix}$. Define the vector function $\mathbf{F}(\mathbf{X})$ by

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} (x-u)(y-2) - (2x-4)(y-v) \\ (x-u)v - (y-v)2u \\ 2(x+2)^2 + (y-2)^2 - 2 \\ u^2 + 2v^2 \end{bmatrix} \ .$$

Then we need to solve $(\mathbf{X}) = \mathbf{0}$.

To choose a starting point. consider the graph that we drew at the beginning. It looks like the point $(1\ ,\ 1/\sqrt{2})$ on the lower ellipse is not too far of from being closest to the upper ellipse. It also looks like the point $(2 - 1/\sqrt{2}\ ,\ 1)$ on the upper ellipse is not too far of from being closest to the lower ellipse.

So we take $\mathbf{X}_0 = \begin{bmatrix} 2 - 1/\sqrt{2} \\ 1 \\ -1 \\ 1/\sqrt{2} \end{bmatrix}$ as our starting point. It is now easy to use a computer to evaluate the successive terms generated by

$$\mathbf{X}_{n+1} = \mathbf{X}_n - [J_{\mathbf{F}}(\mathbf{X}_n)]^{-1}\mathbf{F}(\mathbf{X}_n)\ .$$

Doing the computation with 10 digits, we find that all but the last digits stop changing with 6 iterations, and

$$\mathbf{X}_6 = \begin{bmatrix} -1.422649732 \\ 0.8452994614 \\ -1.154700539 \\ 0.5773502687 \end{bmatrix}\ .$$

Evaluating $f$ at this point, we find $f(\mathbf{X}_6) = 0.1435935399$, and the distance is the square root of this; i.e., 0.378937382... accurate to the digits shown.

The method that we have used in the Example 4 may be used to compute the distance between all sorts of curves. Since we rely on Newton's method to solve the system that we get, we do not have to worry about the algebraic complexity of the equations we get from (2.29).

In the next example, we will consider two constraints in three variables. Then Theorem 2 gives us

$$\nabla f(\mathbf{x}) = \lambda \nabla g_1(\mathbf{x}) + \mu \nabla g_2(\mathbf{x}) \tag{2.30}$$

for some numbers $\lambda$ and $\mu$.

Here is a good way to eliminate $\lambda$ and $\mu$ right at the beginning: Since (2.30) says that $\{\nabla f(\mathbf{x}), \nabla g_1(\mathbf{x}), \nabla g_2(\mathbf{x})\}$ is linearly dependent,

$$\det\left(\begin{bmatrix} \nabla f(\mathbf{x}) \\ \nabla g_1(\mathbf{x}) \\ \nabla g_2(\mathbf{x}) \end{bmatrix}\right) = 0\ . \tag{2.31}$$

This equation, together with $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ gives us three equations in three variables, which is what we want.

**Example 5 (Finding minima and maxima in three variables with two constraints)** As above, consider the problem of maximizing $f(x, y, z) = xyz$ subject to the constraints

$$g_1(x, y, z) = 0$$
$$g_2(x, y, z) = 0$$

where

$$g_1(x, y, z) = x^2 + y^2 + z^2 - 1 \qquad \text{and} \qquad g_2(x, y, z) = x + y + z - 1\ .$$

Then

$$\begin{bmatrix} \nabla f(\mathbf{x}) \\ \nabla g_1(\mathbf{x}) \\ \nabla g_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$$

so that (2.31) reduces to

$$x^2(y-z) + y^2(z-x) + z^2(x-y) = 0 \ .$$

If you go through the algebra carefully – this takes some doing – you will find 6 solutions:

$$(1,0,0) \qquad (0,1,0) \qquad (0,0,1)$$

and

$$(2/3, 2/3, -1/3) \qquad (2/3, -1/3, 2/3) \qquad (-1/3, 2/3, 2/3) \ .$$

At each of the first 3 points $f = 0$, at each of the remaining 3 points $f = -4/9$. Hence the maximum value of $f$ subject to these constraints is 0, and the minimum value is $-4/9$.

In our next example, we will deal with one constraint in three variables. Then Theorem 2 gives us the equation

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \ . \tag{2.32}$$

We can once again eliminate $\lambda$ from the outset. The observation to make is that one vector is a multiple of another if and only if their cross product is zero, Thus, (2.32) is equivalent to

$$\nabla f(\mathbf{x}_0) \times \nabla g(\mathbf{x}_0) = 0 \ .$$

**Example 6 (Finding minima and maxima in three variables with one constraints)** As above, consider the problem of maximizing $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 0$ where

$$g(x,y,z) = x^2 + y^2 + z^2 - 1 \ .$$

Then

$$\nabla f(\mathbf{x}) \times \nabla g(\mathbf{x}) = \begin{bmatrix} x(z^2 - y^2) \\ y(x^2 - z^2) \\ z(y^2 - x^2) \end{bmatrix} \ .$$

Setting this equal to zero would at first seem to give us three equations, but only two of them are independent. Keeping the first two together with the constraint equation gives us the system

$$\begin{aligned} x(z^2 - y^2) &= 0 \\ y(x^2 - z^2) &= 0 \\ x^2 + y^2 + z^2 &= 1 \ . \end{aligned}$$

The first equation says $x = 0$ or $z^2 = y^2$. If $x = 0$, the second equation says $yz^2 = 0$, so then either $y = 0$ or $z = 0$. If $x = y = 0$, the third equation says $x = \pm 1$. Hence if $x = 0$, we have the solutions $(0,0,1)$ and $(0,1,0)$. Otherwise if $z^y = y^2$. If $y = 0$, we get the solution $(1,0,0)$. Otherwise, if $y \neq 0$, we get from the second equation that $x^2 = z^2$ too, so

$$x^2 = y^2 = z^2 \ .$$

Now the third equation says that the common value is $1/3$. Hence we have the nine solutions

$$(1,0,0) \qquad (0,1,0) \qquad (0,0,1)$$

and

$$(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}) \ .$$

From this list we see that the minimum value of $f$ on the unit sphere is $-3^{-3/2}$, and the maximum value is $3^{-3/2}$.

**Exercises**

**1** Consider the two ellipses given by $x^2 + 4y^2 + 3y = 8$ and $(x-5)^2 + 2(y+8)^2 = 2$.

**(a)** Set up a system of 4 equations in 4 variables for determining the pair of closest points on the two ellipses.

**(b)** Draw a graph of the two ellipses, and determine a close approximation to the pair of closest points.

**(c)** Use Newton's method to determine the distance between the two ellipses to at least 6 digits of accuracy.

**2** Consider the hyperbola and the ellipse given by $x^2 - y^2 = 8$ and $x^2 + 2(y+8)^2 = 2$.

**(a)** Set up a system of 4 equations in 4 variables for determining the pair of closest points on the two curves.

**(b)** Draw a graph of the two ellipses, and determine a close approximation to the pair of closest points.

**(c)** Use Newton's method to determine the distance between the two curves to at least 6 digits of accuracy.

**3** For any positive numbers $a$, $b$ and $c$, the volume of the ellipsoid given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

is $(4/3)\pi abc$.

Find the maximum value of this volume, subject to the condition that the point $(1,2,3)$ lies on the boundary of the ellipsoid.

**4** For any positive numbers $a$, $b$ and $c$, the volume of the ellipsoid given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

is $(4/3)\pi abc$.

Find the maximum value of this volume, subject to the condition that the point $(1,2,-1)$ lies on the boundary of the ellipsoid.

**5** The arithmetic–geometric mean inequality states that for any $n$ non negative numbers $x_1, \ldots, x_n$,

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots x_n}{n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. Prove this by determining the maximum value of

$$f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n$$

over the region $D$ given by

$$x_1 + x_2 + \cdots x_n = n$$

and

$$x_j \geq 0 \qquad \text{for all } j\ .$$

**6** Let $D$ be the region consisting of all points $(x,y)$ satisfying

$$x^2 + y^2 + (z-1)^2 \leq 2 \qquad \text{and} \qquad x^2 + y^2 + (z+1)^2 \leq 2\ .$$

Let $f(x,y) = 2x + 3y - z$. Find the minimum and maximum values of $f$ on $D$, and find all minimizers and maximizers. Use Newton's method to solve the equations determining these to at least 6 digits of accuracy if the algebra becomes difficult.

**7** Let $D$ be the region consisting of all points $(x,y)$ satisfying

$$x^4 + y^4 + z^4 = 1 \qquad \text{and} \qquad x^2 + y^2 + xy - z = 0\ .$$

Let $f(x,y) = x + 2y - 3z$. Find the minimum and maximum values of $f$ on $D$, and find all minimizers and maximizers. Use Newton's method to solve the equations determining these to at least 6 digits of accuracy if the algebra becomes difficult.