Math 2605, **First Exam**, 18 Feb 2010, 10am, **NAME:** Solution Key

Neither cheatsheets nor calculators are allowed.

1. (20 points) Find a parametric representation of the line given by the system of equations:

$$\begin{cases} x + 2y + 5z &= 1 \\ 3x - 4y - 5z &= 3 \end{cases}$$

A possible solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 3 & -4 & -5 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & -10 & -20 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

Set $t = z$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ -2t \\ t \end{bmatrix}; \quad \vec{x}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

2. (20 points) Determine whether the line $\boldsymbol{x}(t) = (-2, 3, 2)^T + t(3, -4, 1)^T$ intersects the plane $4x + 4y - 2z = 5$. If so, then at what point?

$$4(3t-2) + 4(-4t+3) - 2(t+2) = 5$$

$$-6t = 5; \quad t = -\frac{5}{6}$$

$$\vec{x}\left(-\frac{5}{6}\right) = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -15/6 \\ 20/6 \\ -5/6 \end{bmatrix} = \left(-\frac{27}{6}, \frac{38}{6}, -\frac{17}{6}\right)^T$$

3. (20 points) Find the distance from the point $(1, 2, -1)$ to the plane $2x + 3y + 5z = 0$.

$\vec{v} = (2,3,5)^T$; $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{38}} \vec{v}$

$\vec{w} = (1,2,-1)^T$ // take $(0,0,0)^T$ as a base point

$\vec{w} \cdot \vec{u} = \boxed{\frac{3}{\sqrt{38}}}$

Find an equation for the plane passing through the given point and parallel to the given plane.

$2(x-1) + 3(y-2) + 5(z+1) = 0$;

$2x + 3y + 5z = 3$.

4. (20 points) Let $f(x, y) = 3x^2 + xy - y^2 - 6xy + 9x - 5y - 2$.

Find the gradient of f,

$$\nabla f = \begin{bmatrix} 6x - 5y + 9 \\ -5x - 2y + 5 \end{bmatrix}$$

the critical points,

$$\left[\begin{array}{cc|c} 6 & -5 & -9 \\ -5 & -2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & -5 & -9 \\ 0 & -\frac{37}{6} & -\frac{15}{6} \end{array}\right] \quad 6x - \frac{75}{37} = -9 \Rightarrow \boxed{x = -\frac{43}{37}}$$

$$\boxed{y = \frac{15}{37}}$$

and Hessian at these points.

$$H_f = \begin{bmatrix} 6 & -5 \\ -5 & -2 \end{bmatrix}$$

For each point determine whether it is a local max, local min, or saddle point.

$D = \det H_f = -37 \Rightarrow$ saddle point.

5. (20 points) Use Lagrange's method to find the maximum and the minimum of

$$f(x, y) = 2x + y^2$$

subject to constraint $x^4 + y^4 = 1$.

$\nabla f = \begin{bmatrix} 2 \\ 2y \end{bmatrix}$; $\nabla g = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}$ (here $g(x,y) = x^4 + y^4 - 1$)

$\det \begin{bmatrix} 2 & 4x^3 \\ 2y & 4y^3 \end{bmatrix} = 8y(y^2 - x^3) = 0$

Case 1: $y = 0 \Rightarrow x^4 = 1 \iff x = \pm 1$

Case 2: $y^2 = x^3 \Rightarrow x^6 + x^4 = 1$ ← This one is hard to solve... but results in $x \approx \pm 0.75$ and produces neither max nor min. No point were taken off for not being able to solve this.

Points: $(1,0)$, $(-1,0)$, $\approx(0.75, 0.75^{\frac{3}{2}})$

$f(1,0) = 2$ is max

$f(-1,0) = -2$ is min

6. (extra credit) Consider the system of equations: $\sin(x) = 0$ and $\sin(y) = 0$. Show that there exists an initial guess $\boldsymbol{x}_0$ such that Newton's method produces a repeating (non-converging) sequence of points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$ such that

$$\boldsymbol{x}_0 = -\boldsymbol{x}_1 = \boldsymbol{x}_2 = -\boldsymbol{x}_3 = \ldots$$

Assume

$-\vec{x} = \vec{x} - J_F^{-1}(\vec{x}) F(\vec{x})$ where $F(\vec{x}) = \begin{bmatrix} \sin x \\ \sin y \end{bmatrix}$

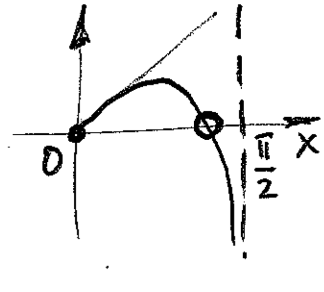
$\begin{bmatrix} 2x \\ 2y \end{bmatrix} = \begin{bmatrix} \cos(x) & 0 \\ 0 & \cos(y) \end{bmatrix}^{-1} \begin{bmatrix} \sin x \\ \sin y \end{bmatrix}$

$\iff \begin{cases} 2x - \tan x = 0 \\ 2y - \tan y = 0 \end{cases}$

$f(x) = 2x - \tan x = 0$ has a solution on the interval $(0, \frac{\pi}{2})$,

since $f(0) = 0$, $f'(0) = 1$, $\lim_{x \to \frac{\pi}{2}} f(x) = -\infty$

Let x_0 be s.t. $f(x_0) = 0$.



Answer: $\vec{x}_0 = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$.