Test 1 for Calculus III for CS Majors, Math 2506 J1-J2, September 23, 2008

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: Consider the function

$$f(x, y) = x^2 + y^2 - 4xy \ .$$

a) (5 points) Calculate the gradient at the point $(1, 1)$.

$$\nabla f(1,1) = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

b) (5 points) Find the line (in parametrized form) that is tangent to the curve $f(x, y) = f(1, 1)$ at the point $(1, 1)$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

c) (5 points) Find the best linear approximation of the function $f(x, y)$ at the point $(1, 1)$.

$$h(x, y) = -2 - 2(x - 1) - 2(y - 1)$$

d) (10 points) Find the points on the curve $f(x, y) = f(1, 1)$ where the tangent line is horizontal.

$$\pm\sqrt{\frac{2}{3}}(2, 1)$$

Problem 2: a) (10 points) Calculate the critical points of the function

$$f(x,y) = 2x^2 + 2y^2 + 3xy - 4x - 3y \ .$$

$$(1,0)$$

b) (10 points) Calculate the Hessian at these critical points.

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

c) (10 points) What are the type of these critical points, are they a maximum a minimum or a saddle?

Minimum

Problem 3: A function $g(x,y)$ with $g(0,0) = 0$ has $(0,0)$ as a critical point and the Hessian at this point is given by

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

a) (5 points)Write the quadratic approximation $q(x,y)$ for the function $g(x,y)$ in the vicinity of this critical point.

$$q(x,y) = \frac{1}{2}(3x^2 + 3y^2 - 4xy)$$

b) (10 points) Find the eigenvalues and the eigenvectors of the Hessian.

Eigenvalue 1 ,Eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvalue 5, Eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c) (10 points) Draw in a qualitative fashion a few of the level curves of $q(x,y)$.

Problem 4: (10 points) a) Set up Newton's scheme for solving the equation $x^2 - y^2 - 1 = 0$ and $2xy - 1 = 0$.

$$\mathbf{x_1} = \mathbf{x_0} - J_F(\mathbf{x_0})^{-1}\mathbf{F}(\mathbf{x_0})$$

b) (10 points) Use as an initial guess the point $\mathbf{x_0} = (1, 0)$ and calculate the next approximation $\mathbf{x_1}$. Check whether this leads to an improvement.

$$\mathbf{F}(\mathbf{x_0}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$J_F(\mathbf{x_0}) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$\mathbf{F}(\mathbf{x_0}) = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix}$$

$$|\mathbf{F}(\mathbf{x_0})| = 1/4$$

improvement

Extra Credit: (15 points) Given the function $f(x, y) = e^x \cos y$ and $g(x, y) = e^y \sin y$. What can you say about the angles between lines tangent to the level curves of f resp. g at any point (x, y)?

$$\nabla f = \begin{bmatrix} 1e^x \cos y \\ -e^x \sin y \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} 1e^x \sin y \\ e^x \cos y \end{bmatrix}$$

$$\nabla f \cdot \nabla g = 0$$