Math 2605
Exam 1

Name:______________________________

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:______________________________

Problem	Possible	Earned
1	10	
2	20	
3	10	
4	10	
5	10	
Total	60	

1. (10 pts) Evaluate the arclength of the curve

$$\mathbf{r}(t) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}, t \right)$$

from the values $0 \leq t \leq 1$.

$$r'(t) = \left(\frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2}, 1 \right)$$

$$\Rightarrow \|r'(t)\|^2 = 1 + \left(\frac{e^t - e^{-t}}{2} \right)^2 + \left(\frac{e^t + e^{-t}}{2} \right)^2$$

$$= 1 + \frac{1}{4} e^{2t} + \frac{1}{4} e^{-2t} + \cancel{\frac{1}{4} \cdot (-2)}$$

$$+ \frac{1}{4} e^{2t} + \frac{1}{4} e^{-2t} + \cancel{\frac{1}{4} \cdot 2}$$

$$= 1 + \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t} = \frac{1}{2} \left(2 + e^{2t} + e^{-2t} \right)$$

$$= \frac{1}{2} \left(e^t + e^{-t} \right)^2$$

$$\Rightarrow \|r'(t)\| = \frac{\sqrt{2}}{2} \cdot \left(e^t + e^{-t} \right)$$

$$\text{Arc Length} = \int_0^1 \|r'(t)\| \, dt$$

$$= \int_0^1 \frac{\sqrt{2}}{2} \left(e^t + e^{-t} \right) dt$$

$$= \frac{\sqrt{2}}{2} \left(e^t - e^{-t} \right) \Big|_0^1 = \boxed{\frac{\sqrt{2}}{2} \left(e - e^{-1} \right)}$$

2. (20 pts) For $t > 0$, find the unit tangent and the principal normal of the function

$$\mathbf{r}(t) = \left(\cos t + t\sin t, \sin t - t\cos t, \frac{\sqrt{3}}{2}t^2\right).$$

Determine the osculating plane to the curve when $t = 0$.

$\hat{r}'(t) = (t\cos t, t\sin t, \sqrt{3}t)$ $\quad \|\hat{r}'(t)\| = 2t \quad (t>0)$

$\Rightarrow \hat{T}(t) = \left(\frac{\cos t}{2}, \frac{\sin t}{2}, \frac{\sqrt{3}}{2}\right)$

$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \frac{\left(-\frac{\sin t}{2}, \frac{\cos t}{2}, 0\right)}{\sqrt{\frac{1}{4}\sin^2 t + \frac{1}{4}\cos^2 t}} = (-\sin t, \cos t, 0)$

$\hat{T}(0) = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \quad \hat{N}(0) = (0, 1, 0)$

$\hat{r}(0) = (1, 0, 0)$

$\hat{T}(0) \times \hat{N}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{vmatrix} = \hat{i}\left(-\frac{\sqrt{3}}{2}\right) - \hat{j}(0) + \hat{k}\left(\frac{1}{2}\right)$

$= \left(-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$

Equation of the plane is $\hat{T}(0) \times \hat{N}(0) \cdot (\hat{X} - \hat{r}(0)) = 0$

$\left(-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) \cdot ((X-1, y, z)) = 0$

$$\boxed{-\frac{\sqrt{3}}{2}(X-1) + \frac{1}{2}z = 0.}$$

3. (10 pts) Find the parametric representation of the line formed by the intersections of the two planes,

$$\begin{aligned} x + 2y + 3z &= 0 \\ -3x + 4y + z &= 0. \end{aligned}$$

Write in matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 4 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Set $z = t$, $y = -z$ $x = -z$

So the line is parameterized by

$$\boxed{\ell(t) = (-1, -1, 1)\, t}$$

4. (10 pts) Let ℓ be the line determined by

$$\ell(u) = (1, -1, 2) + u(-3, 4, -1).$$

Determine where, if at all, the line intersects the plane

$$4x + 4y - 2z = 3.$$

$\ell(u) = (1-3u, -1+4u, 2-u)$

ℓ intersects the plane if and only if

$4(1-3u) + 4(-1+4u) - 2(2-u) = 3$

$\cancel{4} - 12u + (-4) + 16u \cancel{-4} + 2u = 3$

$\Rightarrow 6u = 7$ or $u = \frac{7}{6}$

So the point of intersection is

$\left(1 - 3 \cdot \frac{7}{6}, -1 + \frac{4 \cdot 7}{6}, 2 - \frac{7}{6}\right)$

$= \left(1 - \frac{7}{2}, -1 + \frac{14}{3}, 2 - \frac{7}{6}\right)$

5. (10 pts) Find the distance from the point $(1, 2, 3)$ to the line parameterized by

$$\ell(t) = (1, 0, 2) + t(1, -2, 3).$$

$$\hat{V} = (1, -2, 3) \qquad \hat{X}_0 = (1, 0, 2)$$

$$\hat{p} = (1, 2, 3)$$

$$\hat{p} - \hat{X}_0 = (1, 2, 3) - (1, 0, 2) = (0, 2, 1)$$

$$\hat{V} \to \hat{u} = \frac{(1, -2, 3)}{\|\hat{V}\|} = \frac{(1, -2, 3)}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}(1, -2, 3)$$

Distance from the point to the line ℓ is given by

$$\|(\hat{p} - \hat{X}_0)_\perp\|$$

$$\boxed{(\hat{p} - \hat{X}_0)_\parallel = \frac{1}{\sqrt{14}}(1, -2, 3) \cdot \left(\frac{1}{\sqrt{14}}(-1)\right) = \frac{1}{14}(-1, 2, -3)}$$

$$(\hat{p} - \hat{X}_0)_\perp = (\hat{p} - \hat{X}_0) - (\hat{p} - \hat{X}_0)_\parallel$$

$$= (0, 2, 1) - \frac{1}{14}(-1, 2, -3) = (0, 2, 1) + \frac{1}{14}(1, -2, 3)$$

$$= \left(\frac{1}{14}, 2 - \frac{2}{14}, 1 + \frac{3}{14}\right) = \left(\frac{1}{14}, \frac{26}{14}, \frac{17}{14}\right)$$

$$\|(\hat{p} - \hat{X}_0)_\perp\| = \sqrt{\left(\frac{1}{14}\right)^2 + \left(\frac{26}{14}\right)^2 + \left(\frac{17}{14}\right)^2} =$$