MATH 215 – Winter 2005

SECOND EXAM

Problem 1. (5+5+10=20 points) Throughout this problem f is the function

$$f(x, y) = xye^{-x-2y}.$$

(a) Find all critical points of f.
(b) A calculation shows that

$$f_{xx} = e^{-x-2y}\big(-2y + xy\big), \quad f_{xy} = e^{-x-2y}\big(1 - x - 2y + 2xy\big), \quad f_{yy} = e^{-x-2y}\big(-4x + 4xy\big).$$

Using these formulas, classify the critical points of f. *You do not have to compute the second partials yourself.*

Solution. The partial derivatives are:

$$f_x = e^{-x-2y}(y - xy), \quad f_y = e^{-x-2y}(x - 2y).$$

Since the exponential does not vanish anywhere, the critical points are the solutions to the system

$$y - xy = 0, \quad x - 2y = 0.$$

If $y \neq 0$ then the first equation implies that $x = 1$. If $y = 0$ then x must be zero, by the 2nd equation. Therefore, the critical points are: $(0, 0)$ and $(1, 1/2)$. The Hessian is

$$e^{-x-2y}\begin{pmatrix} -2y + xy & 1 - 2y - x + 2xy \\ 1 - 2y - x + 2xy & -4x + 4xy \end{pmatrix}$$

At $(0, 0)$, the Hessian is

$$e^{1/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and therefore this point is a saddle point.
At $(1, 1/2)$, the Hessian is

$$e^{-2}\begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix}$$

and therefore this point is a local maximum.

□

(c) Find the absolute maximum of the function f on the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1/2)$. Hint: There are not any critical points in the interior of the triangle.

Solution. There are no critical points in the *interior* of the region. We must check the boundary, which consists of 3 sides. On each of the two sides that lie on the coordinate axes, the function is identically zero.

The third side is defined by:

$$x + 2y = 1, \quad 0 \leq x \leq 1.$$

Restricted to this side the function f becomes the one-variable function

$$e^{-1}(1 - 2y)y.$$

The derivative of this function with respect to y is $(1-4y)/e$, which is zero at $y=1/4$. This corresponds to the point $(1/2,1/4)$. Since

$$f(1/2,1/4)=\frac{1}{8e}>0,$$

this must be the absolute maximum of the function.

Conclusion: The absolute maximum value of the function on this region is $f(1/2,1/4)=\frac{1}{8e}$, and the absolute minimum is zero.

□

Problem 2. (15 points) Suppose that the temperature in Celsius at the point (x,y,z) of the ellipsoid $4x^2+y^2+4z^2=6$ is

$$T(x,y,z)=2x+1y-4z+600.$$

Find the hottest and coldest points on the ellipsoid, **and** the highest and lowest values of the temperature.

Solution. Using the method of Lagrange multipliers, one is led to the *system* of equations

$$2=\lambda 8x,\quad 1=\lambda 2y,\quad -4=\lambda 8z$$

plus the constraint. The solutions to this system of equations are:

$$(y=1,\ \lambda=1/2,\ z=-1,\ x=1/2),\quad \text{and}\quad (z=1,\ y=-1,\ \lambda=-1/2,\ x=-1/2).$$

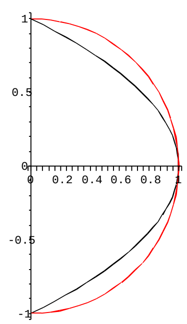
One computes:

$$T(1/2,1,-1)=606,\quad T(-1/2,-1,1)=594.$$

Therefore the hottest point is $(1/2,1,-1)$ with a temperature of 606, and the coldest is $(-1/2,-1,1)$ with a temperature of 594. □

Problem 3. (10+10=20 points) This problem consists of two unrelated questions on double integrals.

(a) Evaluate the double integral $\iint_D xy^2dA$ if D is the region in the half plane $x\geq 0$ bounded by the curves $x=1-y^2$ and $x^2+y^2=1$.



Solution. It is easiest to do the x integral first. Note that in this region the parabola is to the left of the circle, and therefore

$$\iint_D xy^2dA=\int_{-1}^{1}\int_{1-y^2}^{\sqrt{1-y^2}} xy^2\,dx\,dy=\frac{1}{2}\int_{-1}^{1}y^2(1-y^2-(1-y^2)^2)\,dy=\frac{2}{35}.$$

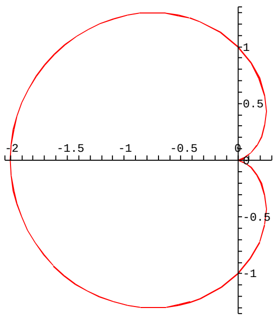
□

(b) Change the order of integration in the iterated integral $\int_0^2 \int_{2x^3}^{8x} f(x,y)\, dy\, dx$. (Do not attempt to evaluate the resulting integral.)

Solution.
$$\int_0^2 \int_{2x^3}^{8x} f(x,y)\, dy\, dx = \int_0^{16} \int_{y/8}^{(y/2)^{1/3}} f(x,y)\, dx\, dy.$$

□

Problem 4. (8+7=15 points) This problem is about finding the the Cartesian coordinates $(\overline{x}, \overline{y})$ of the center of mass of a homogeneous lamina enclosed by the cardioid $r = 1 - \cos(\theta)$ (see figure).



(a) Find the area of the lamina.
(b) Give the value of $\overline{y}$, and express $\overline{x}$ in terms of an iterated double integral where the limits of integration are explicitly determined.

Solution. (a) The area is equal to
$$\iint_D dA = \int_0^{2\pi} \int_0^{1-\cos(\theta)} r\, dr\, d\theta = \frac{1}{2} \int_0^{2\pi} (1 - \cos(\theta))^2\, d\theta = \frac{3\pi}{2}.$$
(b) By symmetry, $\overline{y} = 0$, and.
$$\overline{x} = \frac{2}{3\pi} \int_0^{2\pi} \int_0^{1-\cos(\theta)} r^2 \cos(\theta)\, dr\, d\theta.$$

□

Problem 5. (15 points) In this problem we use the notation:
$$I = \iiint_E f(x,y,z)\, dV,$$
where E is the region in the first octant bounded above by the plane $x + 2y + 3z = 6$.

Express I as an explicit iterated triple integral of the form $\int\int\int dz\, dy\, dx$. Do not attempt to evaluate the integral. Make sure you specify all the limits of integration.

Solution. The limits of the z integration are: $z \geq 0$ since the region is in the first octant, and, solving for z in the equation of the plane, we get $z \leq 2 - x/3 - 2y/3$. The final answer is:
$$I = \int_0^6 \int_0^{3-x/2} \int_0^{2-x/3-2y/3} f\, dz\, dy\, dx.$$

□

Problem 6. (5+10= 15 points) Let E be the region in the first octant in space, inside the sphere $x^2 + y^2 + z^2 = R^2$ and above the cone $3z^2 = x^2 + y^2$.

(a) Find the equation of the cone in spherical coordinates.
(b) Find the volume of the solid.

Solution. (a) Re-writing the equation of the cone in spherical one gets:

$$3\rho^2 \cos(\phi)^2 = r^2 = \rho^2 \sin(\phi)^2.$$

This implies that $3\cos(\phi)^2 = 1 - \cos(\phi)^2$ or $\cos(\phi) = \pm 1/2$. Since ϕ is between 0 and π, the final answer is:

$$\phi = \frac{\pi}{3}.$$

(b) By part (a), the range of ϕ is $0 \leq \phi \leq \pi/3$ (the region is above the cone), Since the region is inside the sphere and above the cone, and in the 1st octant

$$0 \leq \rho \leq R, \quad 0 \leq \theta \leq \pi/2.$$

and the volume is

$$V = \int_0^{\pi/2} \int_0^{\pi/3} \int_0^R \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \frac{\pi R^3}{6} \int_0^{\pi/3} \sin(\phi)\, d\phi = \frac{\pi R^3}{6}(1 - 1/2) = \frac{\pi R^3}{12}$$

□