Test 2 for Calculus III for CS Majors, Math 2506 J1-J2, October 21, 2008

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: Consider the function

$$f(x,y) = x^2 - 6xy + y^2\ .$$

a) (10 points) Find the critical points of $f(x,y)$ inside the domain $x^2+y^2 < 1$ and determine their type, i.e., a local minimum, local maximum or saddle.

$$\nabla f = \begin{bmatrix} 2x - 6y \\ 2y - 6x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yields $(0,0)$ as the only critical point.

The Hessian is

$$H_f(x,y) = \begin{bmatrix} 2 & -6 \\ -6 & 2 \end{bmatrix}$$

Its determinant is negative and hence the critical point is a saddle.

b) (10 points) Find the maximum and minimum of the function $f(x,y)$ on the boundary, i.e., subject to the constraint $x^2 + y^2 - 1 = 0$.

Using Lagrange multiplier we have to solve the equations

$$\begin{bmatrix} 2x - 6y \\ 2y - 6x \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and $x^2 + y^2 = 1$. Cross multiplication yields

$$xy - 3y^2 = xy - 3x^2$$

or $x^2 = y^2$ and the solutions are

$$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), -(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

c) (5 points) Find the maximum and minimum of the function $f(x,y)$ in the region $x^2+y^2 \leq 1$. The ones with opposite signs yield max with max value 4 and the ones with equal sign yield min with min value -2. The critical point in the interior is neither a local max nor local min and hence is not included in the list of candidates for max and min.

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 6 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

a) (5 points) Compute Off(A).

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b) (10 points) Calculate the Givens matrix G for the first step in the Jacobi iteration for diagonalizing A by picking the 2×2 submatrix with the largest off diagonal elements. (You do not have to calculate $G^T AG$.)

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 0 & 1 \\ 0 & \sqrt{10} & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

Another possibility is

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & \sqrt{10} & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

c) (10 points) Calculate Off$(G^T AG)$.

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Problem 3: (10 points) a) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}.$$

$$V = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$D = 3\sqrt{2}$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Assume that

$$V = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}, U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

form the singular decomposition of a matrix B.

b) (10 points) Calculate the generalized inverse B^+

$$A^+ = UD^{-1}V^T = \frac{1}{36}\begin{bmatrix} -3 & -6 & 0 \\ 5 & 2 & 4 \end{bmatrix}$$

c) (5 points) Find the least square solution of the problem $Bx = b$ where

$$b = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$A^+b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 4: a) (10 points) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

b) (10 points) Find a Schur factorization of the matrix A.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

c) (5 points) What is the Schur factorization of A^2?

$$\begin{aligned} A^2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} (\begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix})^2 \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 \\ 0 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Extra Credit: (10 points) Find the Householder reflection that maps the vector $\mathbf{x}$ to the vector $\mathbf{y}$ where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} , \ \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} .$$

$$H = \frac{1}{5} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$