

MATH 215 – Fall 2004
Solutions to the Second Exam

Problem 1. (10+5 = 15 points) This problem is about the iterated double integral

$$I = \int_0^2 \int_{\sqrt{y/2}}^1 y \exp(x^5) dx dy$$

(a) Sketch the region of integration and change the order of integration.

Solution. The domain of integration is the region above the x axis, between the lines $x = 0$ and $x = 1$, and below the parabola $y = 2x^2$. The result of changing the order of integration is:

$$I = \int_0^1 \int_0^{2x^2} y \exp(x^5) dy dx.$$

□

(b) Evaluate I by computing the double integral of your answer to part (a).

Solution. Evaluating the integral with respect to y one obtains:

$$\int_0^{2x^2} y \exp(x^5) dy = \exp(x^5) \int_0^{2x^2} y \exp(x^y) dy = 2 \exp(x^5) x^4,$$

so $I = 2 \int_0^1 x^4 \exp(x^5) dx$. If $u = x^5$, $x^5 dx = du/5$, and so

$$I = \frac{2}{5} \int_0^1 e^u du = \frac{2}{5} (e - 1).$$

□

Problem 2. (10+15=25 points)

(a) Using the method of Lagrange multipliers, find the area of the largest rectangle with pairs of sides parallel to the coordinate axes that can be inscribed in the ellipse $x^2 + 4y^2 = 1$. Also give the coordinates of the corner of the rectangle in the first quadrant.

Solution. If (x, y) denote the coordinates of a point in the first quadrant, the area of the rectangle with a corner at (x, y) , sides parallel to the coordinate axes and center at the origin is: $f(x, y) = (2x)(2y) = 4xy$. The problem is to maximize this function subject to the constraint: $x^2 + 4y^2 = 1$, so that the constraining function is $g(x, y) = x^2 + 4y^2 - 1$.

The vector equation $\nabla f = \lambda \nabla g$ becomes the system:

$$\begin{cases} 4y &= 2\lambda x \\ 4x &= 8\lambda y. \end{cases}$$

Substituting into the 1st equation we get: $2y = 2\lambda^2 y$. Clearly we are not interested in the solution $y = 0$, for then the area is zero. We can therefore divide this equation by y , and get $\lambda = \pm 1$. Since we are looking for a solution with both x, y positive we must have $\lambda = 1$. Substituting $x = 2y$ into the constraint equation one gets: $1 = 4y^2 + 4y^2$, and therefore $y = \frac{\sqrt{2}}{4}$ since $y > 0$. Since $x = 2y$, the final conclusion is:

The coordinates of the corner are: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4})$; the maximal area is 1.

□

(b) Find and classify all the critical points of the function

$$g(x, y) = \sin(x) \cos(y)$$

in the square $[-1, 4] \times [-1, 4]$.

Solution. We compute: $g_x = \cos(x) \cos(y)$, $g_y = -\sin(x) \sin(y)$ and the Hessian at (x, y) is:

$$\begin{vmatrix} -\sin(x) \cos(y) & -\cos(x) \sin(y) \\ -\cos(x) \sin(y) & -\sin(x) \cos(y) \end{vmatrix}$$

so that $D = \sin(x)^2 \cos(y)^2 - \cos(x)^2 \sin(y)^2$.

Let us find critical points. g_x is zero iff at least one of x and y are of the form $\pi/2 + k\pi$, with k an integer. The *only* such value in $[-1, 4]$ is just $\pi/2$; therefore:

$g_x = 0$ in the region iff $x = \pi/2$ or $y = \pi/2$.

Analogously, $g_y = -\sin(x) \sin(y)$. The only values of u in $[-1, 4]$ such that $\sin(u) = 0$ are $u = 0$ and $u = \pi$. Therefore:

$g_y = 0$ in the region iff $(x = 0$ or $x = \pi$ or $y = 0$ or $y = \pi)$.

Since at a critical point *both* partials are zero, there are exactly 4 critical points in the region:

$$P_1 = (\pi/2, 0), P_2 = (\pi/2, \pi), P_3 = (0, \pi/2), P_4 = (\pi, \pi/2).$$

(One coordinate must equal $\pi/2$, and the other can be either 0 or π .) Substituting into the Hessian, one finds:

at $(\pi/2, 0)$ the Hessian is $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$ and therefore P_1 is a local max.

at $(\pi/2, \pi)$ the Hessian is $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ and therefore P_2 is a local min.

at $(0, \pi/2)$ the Hessian is $\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}$ and therefore P_3 is a saddle.

at $(\pi, \pi/2)$ the Hessian is $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and therefore P_4 is a saddle.

□

Problem 3. (5+5= 10 points) The plot below depicts the curve whose equation in polar coordinates is

$$r = 2 - \cos(\theta) :$$

(a) Write an iterated double integral in polar coordinates whose numerical value equals the area enclosed by the curve.

Solution. The region enclosed by the curve is described in polar coordinates by:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2 - \cos(\theta).$$

Therefore, its area is equal to

$$A = \int_0^{2\pi} \int_0^{2-\cos(\theta)} r \, dr \, d\theta.$$

□

(b) Evaluate your answer to (a). The inner integral evaluates to:

$$\int_0^{2-\cos(\theta)} r \, dr = \frac{1}{2}(2 - \cos(\theta))^2 = \frac{1}{2}\left(4 - 4\cos(\theta) + \frac{1 + \cos(2\theta)}{2}\right),$$

where we have used a trig identity for \cos^2 . Since both $\cos(\theta)$ and $\cos(2\theta)$ integrated from zero to 2π give zero,

$$A = \int_0^{2\pi} \frac{1}{2}(4 + 1/2) \, d\theta = \frac{9\pi}{2}.$$

Problem 4. (7+8=15 points) Let E be the region in the first octant bounded by the surfaces $2y^2 + z^2 = 8$ and $x + y = 2$, and let $f(x, y, z)$ be a function whose domain contains E . Denote by

$$I = \iiint_E f(x, y, z) \, dV$$

the integral of f over E .

(a) Set up an iterated triple integral equal to I and of the form

$$I = \iiint f \, dz \, dx \, dy.$$

Solution. The integration is over the region in the first octant above the triangle in the xy plane bounded by the coordinate axes and the line $x + y = 2$, and below the graph $z = \sqrt{8 - 2y^2}$. The answer to part (a) is:

$$I = \int_0^2 \int_0^{2-y} \int_0^{\sqrt{8-2y^2}} f \, dz \, dx \, dy$$

□

(b) Set up an iterated triple integral equal to I and of the form

$$I = \iiint f \, dx \, dy \, dz.$$

Solution. The projection of the region of integration onto the yz plane is the region in the first quadrant of that plane and inside the ellipse $2y^2 + z^2 = 8$. The answer to part (b) is:

$$I = \int_0^{2\sqrt{2}} \int_0^{\sqrt{(8-z^2)/2}} \int_0^{2-y} f \, dx \, dy \, dz$$

□

Problem 5. (10+5=15 points) The problem is to find the Cartesian coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of the region E inside the sphere $x^2 + y^2 + z^2 = R^2$, and between the planes $z = R/2$ and $z = R$.

(a) Give a formula for \bar{z} in terms of an explicit iterated triple integral. You may use without justification the fact that $\text{Vol}(E) = \frac{5}{24}\pi R^3$.

Solution. The region is a solid spherical cap, rotationally symmetric with respect to the z axis. We first note that $\bar{x} = 0 = \bar{y}$, by symmetry. Since we are given the volume of the cap, to compute \bar{z} we must set up an integral for the coordinate z over the region, and then divide that by the volume.

In spherical coordinates: Note that, in the region, $z \geq R/2$. In spherical this translates into:

$$\rho \cos(\phi) \geq \frac{R}{2}.$$

Therefore, in spherical coordinates the region is described by the inequalities:

$$\frac{R}{2 \cos(\phi)} \leq \rho \leq R, \quad 0 \leq \phi \leq \pi/3, \quad 0 \leq \theta \leq 2\pi.$$

The upper bound on ϕ is found by solving the equation: $R \cos(\phi) = R/2$, which is the equation for the intersection of the sphere $\rho = R$ and the plane $z = R/2$. All this leads to:

$$\bar{z} = \frac{24}{5\pi R^3} \int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{R}{2 \cos(\phi)}}^R \rho^3 \cos(\phi) \sin(\phi) d\rho d\phi d\theta.$$

In cylindrical coordinates: In cylindrical the equation of the sphere is: $r^2 + z^2 = R^2$. In addition, the projection of the spherical cap onto the xy plane is the disk centered at the origin and of radius $R\sqrt{3}/2$. Therefore,

$$\bar{z} = \frac{24}{5\pi R^3} \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}R} \int_{R/2}^{\sqrt{R^2-r^2}} z dz r dr d\theta.$$

Other orders of integration in cylindrical are also possible. □

(b) Compute $(\bar{x}, \bar{y}, \bar{z})$.

Solution. As we mentioned already, $\bar{x} = 0 = \bar{y}$ by symmetry. The integral giving \bar{z} turns out to be easier in cylindrical coordinates! Let us evaluate the integral above in cylindrical:

$$\int_{R/2}^{\sqrt{R^2-r^2}} r z dz = \frac{3}{8} R^2 r - \frac{1}{2} r^3.$$

Integrating this with respect to r from zero to $\frac{\sqrt{3}}{2}R$ gives:

$$\int_0^{\frac{\sqrt{3}}{2}R} \int_{R/2}^{\sqrt{R^2-r^2}} z dz r dr = \frac{9}{128} R^4.$$

(This is just a tedious calculation.) The $d\theta$ integration just multiplies this by 2π . After dividing by the volume, we get:

$$\bar{z} = \frac{27}{40} R.$$

(Notice that it's just a little more than $R/2$.) □

Problem 6. (12+8=20 points) The graph below is a plot of some of the level curves of a function f in a rectangular region R . **Assume that the change of the values of the function between adjacent level curves is the same everywhere.** The arrows point in the direction of ∇f (at each point they represent a unit vector in the direction of ∇f).

- (1) On the graph, clearly identify the (approximate) locations of the critical points of f in this region. For each critical point, indicate if it is a local maximum, local minimum or a saddle.

Solution. There are 6 critical points: Two local maxima and a saddle on the left half of the rectangle, and two saddles and a local minimum on the right half. □

- (2) Clearly mark the (approximate) location of the points where this function attains its global maximum and global minimum over the rectangle R .

Solution. The global maximum and minimum are attained at the boundary. This can be seen by counting level curves from the local max/min in the interior. □