Math 215 Second Exam SOLUTIONS *November 13, 2003*

Problem 1. *(15pts.) Find and classify all critical points of the function*

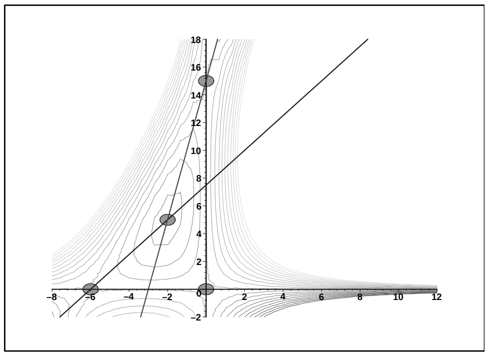
$$f(x,y) = 5x^2y - 2xy^2 + 30xy - 3$$

Solution. First we have to find the critical points by computing the gradient of $f(x,y)$ and setting it to zero:

$$\nabla f = \langle 10xy - 2y^2 + 30y, 5x^2 - 4xy + 30x \rangle = \mathbf{0} \qquad \Longrightarrow \qquad \begin{cases} 2y(5x - y + 15) = 0 \\ x(5x - 4y + 30) = 0 \end{cases}$$

and we see that critical points are intersections of the following lines (see the graph):

$$f_x = 0: \qquad y = 0 \text{ or } 5x - y + 15 = 0$$
$$f_y = 0: \qquad x = 0 \text{ or } 5x - 4y + 30 = 0$$



Plugging in $y = 0$ into the equations for $f_y = 0$ we get two critical points: $(0,0)$ and $(6,0)$. Plugging in $x = 0$ into the second equation for $f_x = 0$ we get the third point $(0,15)$. The remaining point is given by the equations

$$\begin{cases} 5x - y + 15 = 0 \\ 5x - 4y + 30 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} 3y - 15 = 0 \\ 5x = y - 15 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} y = 5 \\ x = -2 \end{cases}$$

and so it is $(-2,5)$. To classify the critical points we will use the second derivative test.
First, we compute the Hessian matrix of second derivatives:

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10y & 10x - 4y + 30 \\ 10x - 4y + 30 & -4x \end{bmatrix}.$$

Then

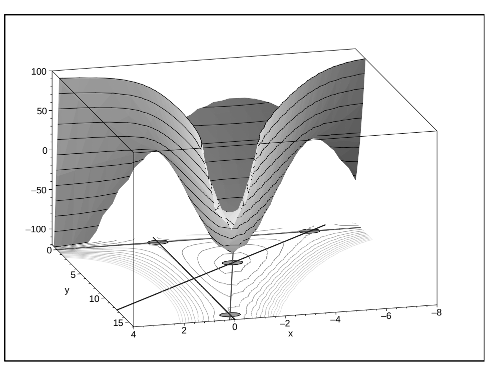
$$H(0,0) = \begin{bmatrix} 0 & 30 \\ 30 & 0 \end{bmatrix} \qquad H(-6,0) = \begin{bmatrix} 0 & -30 \\ -30 & 24 \end{bmatrix} \qquad H(0,15) = \begin{bmatrix} 150 & -30 \\ -30 & 0 \end{bmatrix}$$

and for all these points $\det H < 0$ and so the points are saddles. For the point $(-2,5)$ we have

$$H(-2,5) = \begin{bmatrix} 50 & -10 \\ -10 & 8 \end{bmatrix}, \qquad \det H = 300 > 0, \qquad f_{xx}(-2,5) = 50 > 0$$

and so this point is a local minimum.

Answer: The points $(0,0)$, $(-6,0)$ and $(0,15)$ are *saddles* and the point $(-2,5)$ is a *local minimum*



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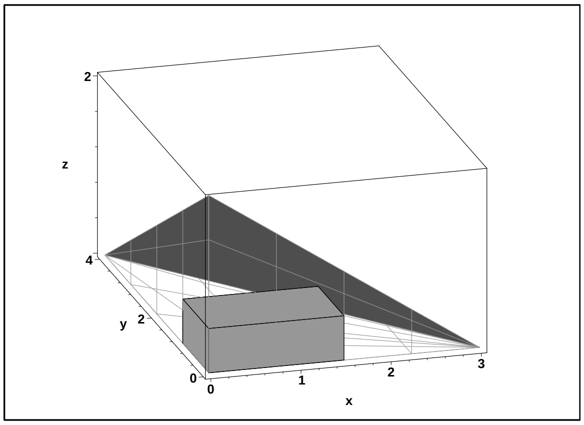
Problem 2. *(10 pts.) Use Lagrange multipliers to determine the dimensions of the storage box of the largest volume that can be pushed into the remote dusty corner of the attic as shown on the picture below.*



Solution. From the picture we see that the box is required to lie in the first octant below the plane representing the roof. Since the intersections of this plane with the coordinate axes are $x = 3$, $y = 4$, and $z = 2$, the equation of the plane is $4x+3y+6z = 12$ (or $z = 2 - 2/3x - 1/2y$) and we arrive at the following constrained optimization problem:

$$\text{Volume} = f(x, y, z) = xyz \to \text{max subject to the constraint } g(x, y, z) = 4x + 3y + 6z = 12.$$

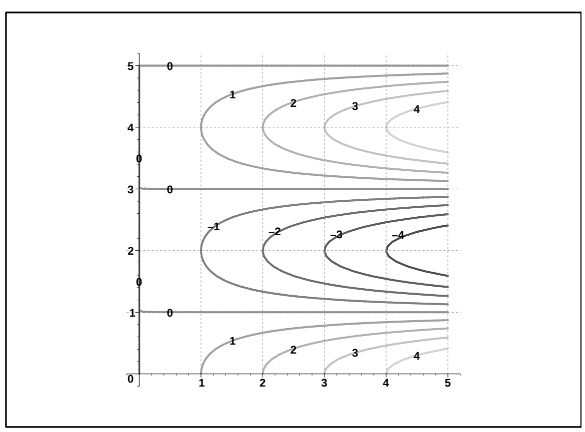
The Lagrange multiplier equations then are

$$\begin{cases} yz = 4\lambda \\ xz = 3\lambda \\ xy = 6\lambda \\ 4x + 3y + 6z = 12 \end{cases} \implies \begin{array}{c} xyz = 4\lambda x = 3\lambda y = 6\lambda z \\ y = 4/3x, \quad z = 4/6x \end{array} \implies 4x + 4x + 4x = 12$$

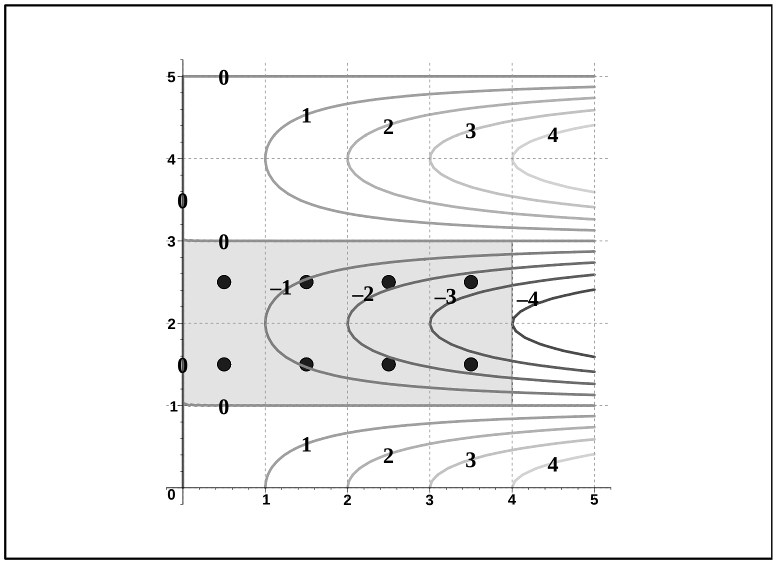
and so the dimensions of the box are $\boxed{x = 1, y = 4/3, z = 2/3}$ and the maximal volume is $8/9$. Note that in solving these equations we implicity assumed that all dimensions are non-zero. In fact, there are solutions when two of the sides and λ are zero, but this clearly gives a flat box of minimum zero volume. ♣

Problem 3. *(10 pts) Consider the function* $f(x, y) = x\cos\left(\frac{\pi y}{2}\right)$ *over the rectangle* $R = [0, 4] \times [1, 3]$.

(a–5 pts.) Using the level curves of this function on the plot below, estimate $\iint_R f(x, y)\, dA$ *(you have to* ***carefully explain*** *your setup).*



Solution. Using subdivision given by the coordinate lines (and so $\Delta A_{ij} = 1$) and using the mid-points for sampling as shown on the picture below,



we get

$$\iint_R f(x,y)\,dA \approx (-0.3 - 1.1 - 1.7 - 2.3 - 0.3 - 1.1 - 1.7 - 2.3) = \boxed{-10.8}$$

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(b–5 pts.) Find the exact value of this integral.

Solution.

$$\iint_R x\cos\left(\frac{\pi y}{2}\right)\,dA = \int_0^4 \int_1^3 x\cos\left(\frac{\pi y}{2}\right), dy\,dx = \int_0^4 x\,dx \int_1^3 \cos\left(\frac{\pi y}{2}\right)\,dy$$

$$= \frac{4^2}{2}\cdot\frac{2}{\pi}\cdot\sin\left(\frac{\pi y}{2}\right)\Big|_1^3 = \boxed{-\frac{32}{\pi}} \approx -10.18,$$

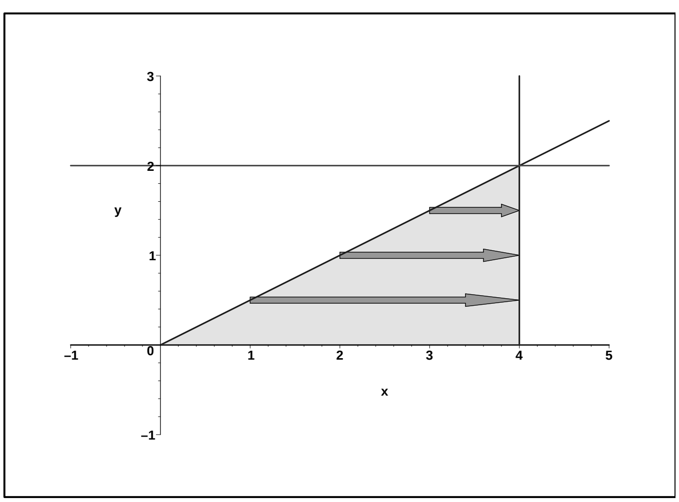
and we see that our Riemann Sum approximation above was pretty close. ♣

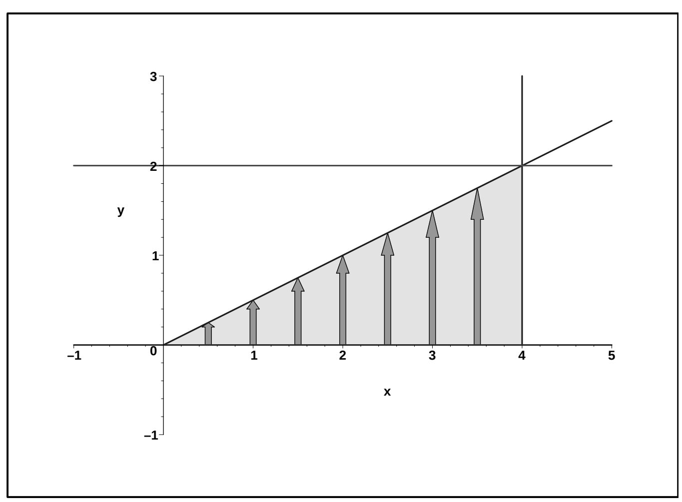
Problem 4. *(10 pts.) Sketch the region of integration and evaluate the integral*

$$\int_0^2 \int_{2y}^4 \sin x^2\,dx\,dy$$

by changing the order of integration.

Solution.





$$\int_0^2 \int_{2y}^4 \sin x^2\,dx\,dy = \iint_D \sin x^2\,dA = \int_0^4 \int_0^{\frac{x}{2}} \sin x^2\,dy\,dx = \int_0^4 \frac{x}{2}\sin x^2\,dx = -\frac{1}{4}\cos x^2\Big|_0^4 = \boxed{\frac{1-\cos 16}{4}}$$

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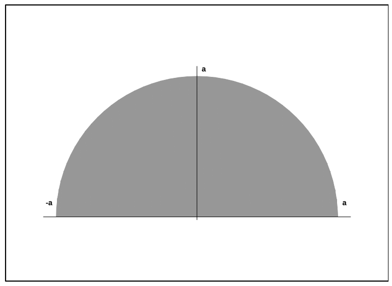
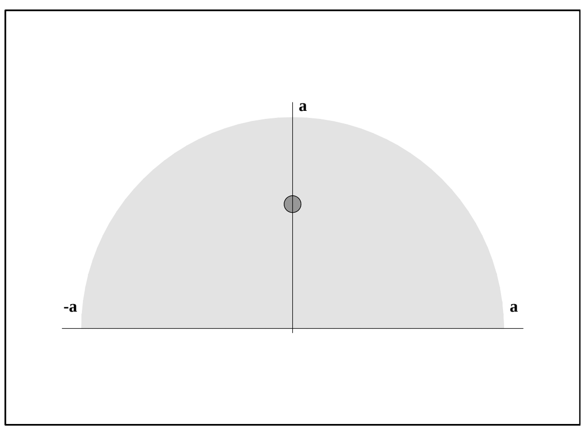
Problem 5. *(15 pts.) Find the center of mass of a non-homogeneous lamina in the shape of a semi-circle of radius a cm (see the picture below) if the density function is $\sigma(x, y) = y$ g/cm^2 .*



Solution. Using polar coordinates, we get

$$\text{Mass} = \iint_D \sigma(x,y)\,dA = \int_0^\pi \int_0^a (r\sin\theta) r\,dr\,d\theta = 2\frac{a^3}{3}$$
$$M_y = \iint_D x\sigma(x,y)\,dA = \iint_D xy\,dA = 0 \text{ and so } \bar{x} = 0 \qquad \text{(by symmetry)}$$
$$M_x = \iint_D y\sigma(x,y)\,dA = \int_0^\pi \int_0^a (r\sin\theta)^2 r\,dr\,d\theta = \int_0^\pi \sin^2\theta\,d\theta \cdot \int_0^a r^3\,da$$
$$= \frac{a^4}{4} \cdot \int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\pi a^4}{8} \text{ and so } \bar{y} = \frac{M_x}{\text{Mass}} = \frac{3\pi a}{16}.$$

Answer: The center of mass is $\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{3\pi a}{16}\right)}$.



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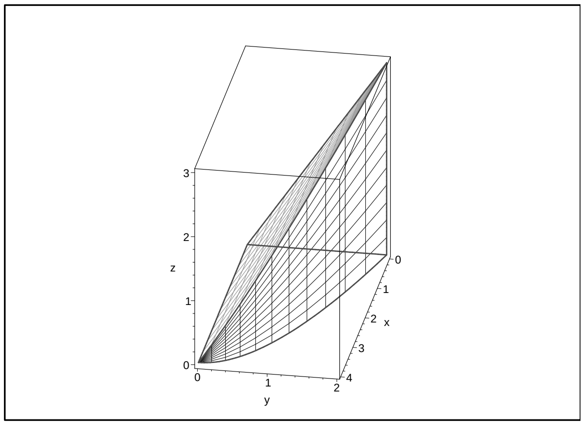
Problem 6. *(15 pts.) Consider the iterated triple integral*

$$\int_0^3 \int_{2z/3}^2 \int_0^{4-y^2} y\,dx\,dy\,dz$$

(a–10 pts.) Sketch the region of integration E and then rewrite this integral as $\iiint_E y\,dz\,dx\,dy$, i.e., find the appropriate limits of integration (note that your sketch can be helpful).

Solution. The region of integration a part of the first octant bounded by the coordinate planes, the plane $y = 2z/3$ and the cylinder $x = 4 - y^2$. Sketching the region of integration,



we see that for any point in the xy-plane the limits of integration in the z-variable are from $z = 0$ to the plane; solving the equation of the plane for z we get

$$\int_0^3 \int_{2z/3}^2 \int_0^{4-y^2} y \, dx \, dy \, dz = \iiint_E y \, dV = \int_0^2 \int_0^{4-y^2} \int_0^{3y/2} y \, dz \, dx \, dy.$$

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(b–5 pts.) Evaluate this integral.

Solution. We can compute this integral in both ways of integration,

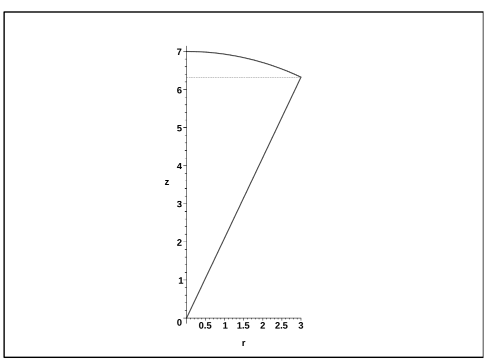
$$\int_0^2 \int_0^{4-y^2} \int_0^{3y/2} y \, dz \, dx \, dy = \int_0^2 y(4-y^2)(3y/2) \, dy = \int_0^2 \left(6y^2 - \frac{3}{2}y^4\right) dy = 2 \cdot 2^3 - \frac{3}{2}\frac{1}{5}2^5 = \frac{32}{5} = \boxed{6.4}$$

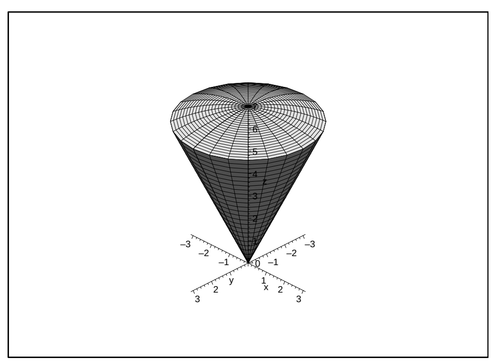
$$\int_0^3 \int_{2z/3}^2 \int_0^{4-y^2} y \, dx \, dy \, dz = \int_0^3 \int_{2z/3}^2 (4y - y^3) \, dy \, dz \int_0^3 \left(8 - \frac{8z^2}{9} - 4 + \frac{4z^4}{3^4}\right) dz = 4 + \frac{12}{5} = 6.4,$$

but note that the first integral is quite a bit easier to compute than the second one. ♣

Problem 7. *(15 pts.) Find the amount (mass) of ice-cream in the ice-cream cone formed by a sphere of radius* 7 cm *centered at the origin and a cone opening upwards from the origin with top radius of* 3 cm *if the density of ice-cream is given by* $\sigma(x, y, z) = z$ g/cm^3 *(which makes sense, since the ice-cream is more compressed near the spoon and is more loose towards the bottom of the cone).*

Solution.





This question is best done in spherical coordinates. Denoting by Φ the angle the cone makes with the z-axis ($\sin\Phi = 3/7$), we get

$$\text{Mass} = \iiint_E z \, dV = \int_0^{2\pi} \int_0^{\Phi} \int_0^7 (\rho\cos\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta = 2\pi \int_0^7 \rho^3 \, d\rho \cdot \int_0^{\Phi} \cos\phi \sin\phi \, d\phi$$

$$= 2\pi \cdot \frac{7^4}{4} \cdot \frac{\sin^2\phi}{2} = 2\pi \cdot \frac{7^4}{4} \cdot \frac{\frac{3^2}{7^2}}{2} = \pi \cdot \frac{7^2 \cdot 3^2}{4} = \frac{441\pi}{4}$$

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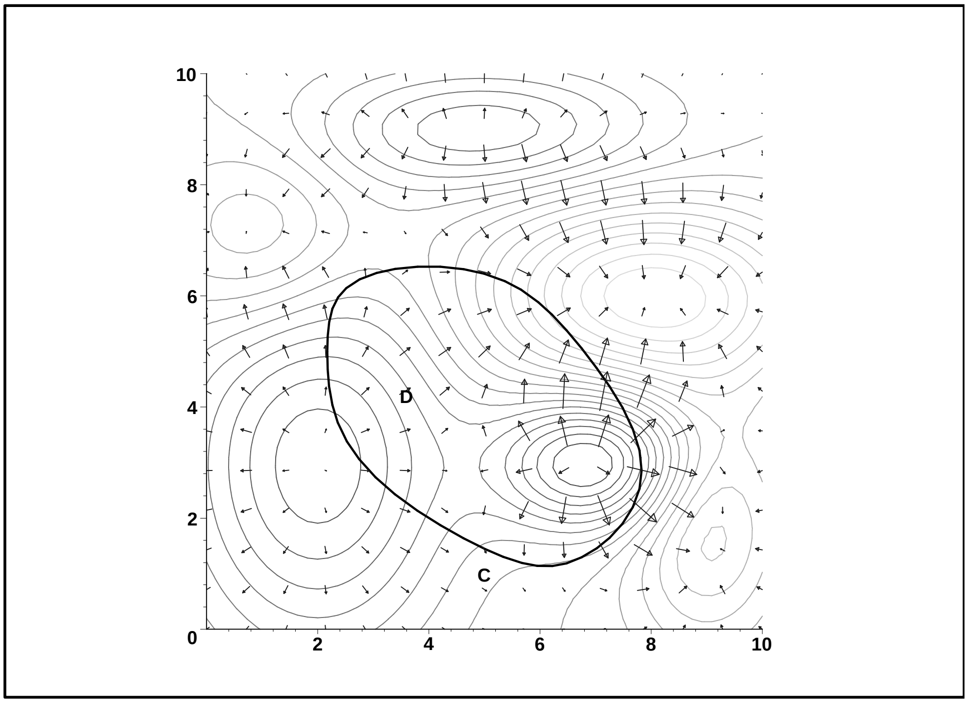
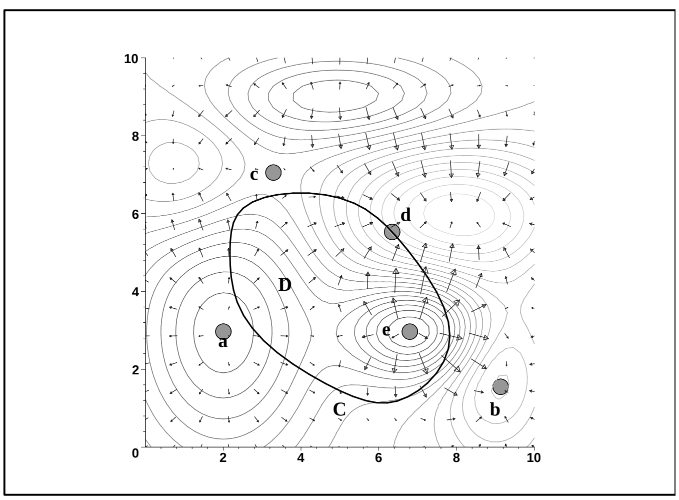
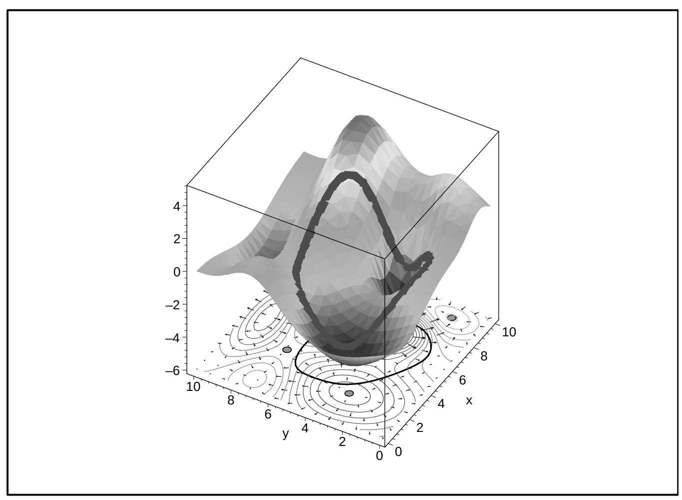
Problem 8. *(10 pts.) The graph below is a plot of some level curves for a function* $f(x, y)$*, along with arrows representing the gradient* $\nabla(f)$ *(adjacent level curves represent the same change in* f*).* D *is the region bounded by (and including) the oval constraint curve* C*. On the graph, please carefully mark and label the following:*

(a – 2 pts.) An example of a critical point that is a local minimum.
(b – 2 pts.) An example of a critical point that is a local maximim.
(c – 2 pts.) An example of a critical point that is a saddle.
(d – 2 pts.) Location of the absolute maximum of $f(x,y)$ on the region D.
(e – 2 pts.) Location of the absolute minimum of $f(x,y)$ on the region D.



Solution. See the graphs below (note that parts (a–c) allow other solutions):





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