Math 2605, **Third Exam**, 22 Apr 2010, 10am, **NAME:**

Neither cheatsheets nor calculators are allowed.

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1. (20 points) Establish a Schur decomposition $A = QTQ^T$ with an upper-triangular T and orthogonal Q for

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.$$

Find e-value, e-vector.

$$\begin{vmatrix} 1-\mu & 1 \\ -1 & 3-\mu \end{vmatrix} = \mu^2 - 4\mu + 4 = (\mu-2)^2 = 0;$$

$$\mu = 2; \quad \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\vec{v} = 0; \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Make an orthogonal matrix

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note: $\vec{u}_1 \perp \vec{u}_2$

$$Q = [\vec{u}_1, \vec{u}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad T = Q^T A Q =$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

2. (20 points) Find a Householder reflection M mapping $\boldsymbol{x} = (3i, 4)^T$ onto a multiple of $\boldsymbol{e}_1$.

$\|\vec{x}\| = 5$. Try $\langle \vec{x}, 5\vec{e}_1 \rangle = 15i$

Need $\langle \vec{x}, \vec{y} \rangle \in \mathbb{R}$: choose $\vec{y} = 5i\,\vec{e}_1$

$$\vec{u} = \frac{\vec{x} - \vec{y}}{\|\vec{x} - \vec{y}\|} = \frac{1}{\sqrt{20}}\begin{bmatrix} -2i \\ 4 \end{bmatrix}; \quad \vec{u}\vec{u}^* = \frac{1}{20}\begin{bmatrix} 4 & -8i \\ 8i & 16 \end{bmatrix}$$

$$M = I - 2\vec{u}\vec{u}^* = \frac{1}{10}\left[\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 4 & -8i \\ 8i & 16 \end{bmatrix}\right] =$$

$$= \frac{1}{10}\begin{bmatrix} 6 & 8i \\ -8i & -6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4i \\ -4i & -3 \end{bmatrix}$$

Note: $M^* = M$

3. (20 points) Describe the set of *unit* vectors in $\mathbb{C}^2$ orthogonal to $\boldsymbol{v} = (1, i)^T$.

The vector $\vec{u} = \frac{1}{\sqrt{2}}(i, 1)^T$ is one such vector.

The set consists of $\alpha\vec{u}$ with $|\alpha| = 1$.

4. (20 points) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Its QR decomposition is

$$A = QR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

Compute the matrix $A^{(1)}$ obtained after one QR iteration.

$$A^{(1)} = RQ = \left(\frac{1}{\sqrt{2}} \cdot \sqrt{2}\right) \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}$$

Let μ be an eigenvalue of A. Prove that μ is also an eigenvalue of the similar matrix $B^{-1}AB$ for any invertible B.

Let $\vec{v}$ be an e-vector s.t. $A\vec{v} = \mu\vec{v}$, then $B^{-1}\vec{v}$ ~~is an e-vector~~ is also an e-vector for $B^{-1}AB$: indeed, $B^{-1}AB(B^{-1}\vec{v}) = B^{-1}A\vec{v} = \mu(B^{-1}\vec{v})$.
The e-value corresponding to $(B^{-1}\vec{v})$ is μ.

~~What are the eigenvalues of A? Explain.~~

5. (20 points) Using the exponential function of a matrix find the solutions $(x(t), y(t))$ of the following system of ordinary differential equations:

$$\begin{cases} x' &= x + 2y \\ y' &= 2y \end{cases}$$

such that $x(0) = 1$ and $y(0) = 2$.

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$; initial cond: $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution: $\vec{x}(t) = e^{tA}\vec{x}_0$.

Since A is upper-Δ, $e^{tA} = \begin{bmatrix} e^t & a \\ 0 & e^{2t} \end{bmatrix}$,

where $a = 2t \dfrac{e^{2t} - e^t}{2t - t} = 2(e^{2t} - e^t)$.

$$\vec{x}(t) = \begin{bmatrix} e^t + 4(e^{2t} - e^t) \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} -3e^t + 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

6. (extra credit, use the back of the page if necessary) Let matrix

$$A = \begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & x \end{pmatrix}$$

Here: $\binom{n}{k}$ are Binomial coeff's

What is A^4? A^n? (Hint: $A = xI + U$)

$$u = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u^3 = \dots, \quad u^4 = \dots, \quad u^5 = \begin{bmatrix} \text{zero} \\ \text{matrix} \end{bmatrix}$$

$$(xI + u)^4 = \begin{bmatrix} x^4 & 4x^3 & 6x^2 & 4x & 1 \\ & x^4 & 4x^3 & 6x^2 & 4x \\ & & x^4 & 4x^3 & 6x^2 \\ & 0 & & x^4 & 4x^3 \\ & & & & x^4 \end{bmatrix}, \quad A^n = \begin{bmatrix} x^4 & nx^3 & \binom{n}{2}x^2 & \binom{n}{3}x & \binom{n}{4} \\ & & & & \\ 0 & & & & \end{bmatrix}$$