Math 2605, **Third Exam**, 22 Apr 2010, 3pm, **NAME:**

Neither cheatsheets nor calculators are allowed.

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1. (20 points) Establish a Schur decomposition $A = QTQ^T$ with an upper-triangular T and orthogonal Q for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

$$\begin{vmatrix} 1-\mu & -1 \\ 1 & 3-\mu \end{vmatrix} = \mu^2 - 4\mu + 4 = (\mu-2)^2 = 0;$$

$$\mu = 2; \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{v} = 0; \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: $\vec{u}_1 \perp \vec{u}_2$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad T = Q^T A Q = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

2. Find a vector $\boldsymbol{w}$ in $\mathbb{C}^2$ that is orthogonal to $u = (7 + 6i, 6)^T$.

$$\vec{w} = (-6, 7-6i)^T$$

Find a 2×2 unitary matrix whose first column is a real multiple of u.

$$\|\vec{w}\| = \|\vec{u}\| = \sqrt{36+36+49} = 11.$$

$$U = \frac{1}{11} \begin{bmatrix} 7+6i & -6 \\ 6 & 7-6i \end{bmatrix}.$$

3. (20 points) Find the *good* Householder reflection M mapping $\boldsymbol{x} = (3,4)^T$ onto a multiple of $\boldsymbol{e}_1$.

$\|\vec{x}\| = 5 \qquad \vec{y} = -5\vec{e}_1.$

$$\vec{u} = \frac{\vec{x}-\vec{y}}{\|\vec{x}-\vec{y}\|} = \frac{1}{\sqrt{80}}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{u}\vec{u}^T = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$M = I - 2\vec{u}\vec{u}^T = \frac{1}{5}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$$

4. (20 points) (1) Prove that an eigenvector of A is an eigenvector of $A - aI$ for any number a.

Let $A\vec{v} = \mu\vec{v}$. Then $(A - aI)\vec{v} = A\vec{v} - a\vec{v}$
$= \mu\vec{v} - a\vec{v} = (\mu - a)\vec{v}.$

This shows that $\vec{v}$ is an e-vector of $A - aI$ with e-value $\mu - a$.

(2) One way to speedup the QR iteration algorithm for finding an eigenvector of A is to consider $B = (A - aI)^{-1}$ where a is a rough approximation for an eigenvalue μ of A.

- If $\boldsymbol{v}$ is an eigenvector of A corresponding to μ, what does λ equal in $B\boldsymbol{v} = \lambda\boldsymbol{v}$?

$$\lambda = \frac{1}{\mu - a}$$

- Explain why a speedup is expected.

λ is large if a is close to μ.

5. (20 points) Given the Schur decomposition $A = UTU^T$, what is e^A in terms of e^T?

$e^A = Ue^TU^T.$

Suppose in the above decomposition

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Find the exponential e^A.

$e^T = \begin{pmatrix} e & a \\ 0 & e^3 \end{pmatrix}$, where $a = \frac{2(e^3-e)}{3-1} = e^3 - e.$

$$e^A = Ue^TU^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} e & e^3-e \\ 0 & e \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} e^3 & e^3-2e \\ e^3 & e^3 \end{bmatrix}} = \begin{bmatrix} e^3 & e^3-e \\ 0 & e \end{bmatrix}$$

$$\begin{bmatrix} 2e^3 & 2e^3-2e \\ 0 & 2e \end{bmatrix}$$

6. (extra credit, use the back of the page if necessary) Let matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

What is A^4? A^n? (Hint: $A = I + U$)

Here: $\binom{n}{k}$ are Binomial coeff's

$$U = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}, \; U^2 = \begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}, \; U^3 = \dots, \; U^4 = \dots, \; U^5 = \begin{bmatrix} \text{zero} \\ \text{matrix} \end{bmatrix}$$

$$(I+U)^4 = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ & 1 & 4 & 6 & 4 \\ & & 1 & 4 & 6 \\ & 0 & & 1 & 4 \\ & & & & 1 \end{bmatrix}, \qquad A^n = \begin{bmatrix} 1 & n & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & & & & \end{bmatrix}$$