Math 2605 Quiz 2

21 Jan 10

Name:

Consider two lines in $\mathbb{R}^3$ given parametrically by $l_1 : \mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $l_2 : \mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

1. (5 points) Compute the distance between these two lines.
2. (5 points) Write a parametric representation of the line that passes through the two points where l_1 and l_2 are closest to each other.

Solution:

1. $l_1: \begin{pmatrix} 1+s \\ 2+2s \\ 3+2s \end{pmatrix}$, $l_2: \begin{pmatrix} 2-2t \\ t \\ 2+t \end{pmatrix}$

$d^2(s,t) = (1+s-2+2t)^2 + (2+2s-t) + (3+2s-2-t)^2$

fix t, the derivative of $d^2(s,t)$ w.r.t. s is

$18s - 4t + 10$

fix s, the derivative of $d^2(s,t)$ w.r.t t is

$-4s + 12t - 10$

Solve the L.S.E

$$\begin{cases} 18s - 4t + 10 = 0 \\ -4s + 12t - 10 = 0 \end{cases}$$

We have $s = -\frac{2}{5}$, $t = \frac{7}{10}$, so

$d(-\frac{2}{5}, \frac{7}{10}) = \frac{1}{\sqrt{5}}$ is the shortest distance between the two lines.

2. The two points where the two lines are the nearest to each other are:

$$\begin{pmatrix} \frac{3}{5} \\ \frac{6}{5} \\ \frac{11}{5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{3}{5} \\ \frac{7}{10} \\ \frac{27}{10} \end{pmatrix}$$

so the line passing through these two points is

$$\begin{pmatrix} \frac{3}{5} \\ \frac{6}{5} \\ \frac{11}{5} \end{pmatrix} + u \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Another solution:

Let $A = [v_1, -v_2]$ and $b = x_2 - x_1$, then s.t satisfies

$$A^t A \begin{pmatrix} s \\ t \end{pmatrix} = A^t b$$

that is

$$\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{7}{10} \end{pmatrix}$$

All the rest are the same ....