Math 2605-C Quiz 3 **Name:** SOLUTIONS

28 Jan 10

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \end{cases} \quad \text{and} \quad f(0,0) = 0$$

1. (5 points) Calculate $\frac{\partial f}{\partial x}(5,0)$and $\frac{\partial f}{\partial y}(5,0)$.
2. (5 points) Show that f is not continuous at the point $(0,0)$.

① write $f(x,y) = xy(x^2+y^2)^{-1/2}$

then $\frac{\partial f}{\partial x} = y(x^2+y^2)^{-1/2} + xy\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2x)$

$= y(x^2+y^2)^{-1/2} - x^2y(x^2+y^2)^{-3/2}$

By symmetry:

$\frac{\partial f}{\partial y} = x(x^2+y^2)^{-1/2} - xy^2(x^2+y^2)^{-3/2}$

To evaluate, plug in 5 for x and 0 for y:

$\frac{\partial f}{\partial x}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = 0$ $\qquad$ $\frac{\partial f}{\partial y}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{5}{\sqrt{25}} = 1$

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(2) there was a TYPO. Sorry about that!

The function as written IS continuous at $(0,0)$

Here is a proof:

We start by observing that

$(x+y)^2 = x^2 + 2xy + y^2$, and

$(x-y)^2 = x^2 - 2xy + y^2$, so that

$$4xy = (x+y)^2 - (x-y)^2 \quad (*)$$

In the first & third quadrants, when x & y have the same sign:

$$4xy \leq (x+y)^2 = x^2 + 2xy + y^2,$$

so $xy \leq \frac{1}{2}(x^2+y^2)$

$$\Rightarrow \frac{xy}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\sqrt{x^2+y^2}$$

so as $\sqrt{x^2+y^2} \to 0$, $|f(x,y)| \to 0$

in the 2nd and 4th quadrants,

xy is negative.

Using $(*)$, write

$$4xy \geq -(x-y)^2 = -x^2 + 2xy - y^2$$

so $xy \geq -\frac{1}{2}(x^2+y^2)$

$$\Rightarrow \quad -\frac{xy}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\sqrt{x^2+y^2}$$

putting things together,

$$|f(x,y)| \leq \frac{1}{2}\sqrt{x^2+y^2},$$

so as $x^2+y^2 \to 0$, $|f| \to 0$

since $f(0,0) = 0$ by definition,

f is continuous at $(0,0)$.

the function we <u>meant</u> to give:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^4+y^4}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

this function is <u>larger</u> near the origin (its denominator is smaller)

and it's <u>not</u> continuous. <u>Proof</u>:

Let $y = x$. Along this line,

$$f(x,y) = f(x,x) = \frac{x^2}{\sqrt{2x^4}} = \frac{1}{\sqrt{2}} \neq 0$$

Since there are points on the line $y=x$ arbitrarily close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, f is not continuous there!