Math 2605-C Quiz 5 **Name:** SOLUTION
11 Feb 10

Consider the function

$$f(x, y) = (x^2 + y^2)e^{x^2-y^2}$$

1. (10 points) Identify the critical points and determine whether each is a local maximum, local minimum, or saddle point.

   (Hint: there are three such points. Be patient and make sure to take the partial derivatives correctly.)

$$\frac{\partial f}{\partial x} = 2x(e^{x^2-y^2}) + 2x(x^2+y^2)e^{x^2-y^2}$$

$$= 2x \cdot e^{x^2-y^2}(1 + x^2 + y^2)$$

this is a product of three functions, but two of them are never equal to 0!

so $\frac{\partial f}{\partial x} = 0$ only when $\boxed{2x = x = 0}$

Now, critical points are where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both vanish, so we'll compute $\frac{\partial f}{\partial y}$ and then plug in $\boxed{x = 0}$

$$\frac{\partial f}{\partial y} = 2y\, e^{x^2-y^2}(1-x^2-y^2)$$

Notice that you can use the symmetry of the function to avoid calculating this "from scratch." The only thing that changes is the sign in the second term and replacing x with y.

when $x = 0$ we get:

$$\frac{\partial f}{\partial y}(0,y) = 2y\, e^{-y^2}(1-y^2)$$

this is zero when $y = 0, 1, -1$

so our critical points are:

$$\boxed{\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Now we need to compute the second partials. It will be helpful to rewrite the first partials.

$$\frac{\partial f}{\partial x} = e^{x^2-y^2}(2x + 2x^3 + 2xy^2)$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = 2x\,e^{x^2-y^2}(2x + 2x^3 + 2xy^2) + e^{x^2-y^2}(2 + 6x^2 + 2y^2)}$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = -2y\,e^{x^2-y^2}(2x + 2x^3 + 2xy^2) + e^{x^2-y^2}(4xy)}$$

Similarly, write

$$\frac{\partial f}{\partial y} = e^{x^2-y^2}(2y - 2x^2y - 2y^3)$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = -2y\,e^{x^2-y^2}(2y - 2x^2y - 2y^3) + e^{x^2-y^2}(2 - 2x^2 - 6y^2)}$$

Note: 1 the mixed partials are equal

2 we could simplify ~~thiese~~ these expressions but we don't need to

recall:

$$H_f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Alternative notation, may be easier to remember:

$$H_f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Now we test our points:

① $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

determinant is positive and ~~and~~ f_{xx} and f_{yy} are positive, so this is a $\boxed{\text{minimum (local min)}}$

(Analyzing the eigenvalues reveals that this point has a constant curvature equal to 2)

(2) $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \quad H_f = \begin{bmatrix} 4/e & 0 \\ 0 & -4/e \end{bmatrix}$

determinant is

$-16/e^2 < 0$, so this is a $\boxed{\text{saddle point}}$

(3) $\begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \quad H_f = \begin{bmatrix} 4/e & 0 \\ 0 & -4/e \end{bmatrix}$

so also a $\boxed{\text{saddle point}}$