Frenet Frames of Curves

We studied vector valued parametric curves in three dimensions in class. If the curve r(t) is parameterized by time t with t belonging to the interval [a,b], then we saw in class that the tangent to the curve is given by r'(t) and points in the direction of motion along the curve.

It was shown that the unit tangent is given by:

$$T(t) := \frac{r'(t)}{\|r'(t)\|}$$

We also showed that the curve since T has unit norm that $T'(t) \cdot T(t) = 0$, so that the derivative of the unit tangent is orthogonal to the vector T(t). Using this we then defined the function

$$N(t) := \frac{T'(t)}{\|T'(t)\|}$$

We then defined the binormal

$$B(t) := \frac{T(t) \times N(t)}{\|T(t) \times N(t)\|}.$$

The osculating plane is defined as the plane spanned by the vectors, T and N. The normal plane is the plane spanned by B and N, while the rectifying plane is the one spanned by B and T. These planes are describing the "geometry" of the curve as it moves through out space.

The osculating circle of a sufficiently smooth curve at a given point on the curve is defined to be the circle whose center lies on the inner normal line and whose curvature is the same as that of the given curve at that point. The oscullating circle will of course lie in the oscullating plane.

After computing these vectors, we can see that there is a natural relationship between them that is encoded by the curvature and torsion of the parametric curve. The curvature is measuring how much the curve is bending as it is traced out, while the torsion is measuring the twisting of the curve. The curvature and torsion of a curve in three dimensions are defined via the following formulas. Namely,

$$T' = \kappa N$$

Where κ is the curvature of the curve at the point $r(t)$. One can show that the curvature is give by:

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$$

The torsion τ measures the speed of rotation of the binormal vector at the given point. It is found from the equation

$$B' = -\tau N$$

which means

$$\tau = -N \cdot B'.$$

One can further show that the functions T, N, B must satisfy the following relationship:

$$\begin{aligned} T' &= \|r'(t)\|\kappa N \\ N' &= \|r'(t)\| \left(-\kappa T + \tau B\right) \\ B' &= -\|r'(t)\|\tau N \end{aligned}$$

So, for each value of t in the interval $[a, b]$, to the point $r(t)$ we have associated a frame of vectors T(t), N(t), B(t). This frame of vectors is called the Frenet-Serret frame. The curvature and torsion demonstrate and measure the bending and twisting of the curve. While the oscullating plane, normal plane, and rectifying plane provide the tangent planes with the greatest amount of contact (in the appropriate directions). The geometry of the curve is controlled by the frame of vectors, the curvature and the torsion.

Your job in this project is to write an algorithm that will read in a curve in three dimensions. Then compute some of these quantities discussed above. The program should provide as output a graph of these curves and objects.

1. Graph the curve r(t).
2. Write and algorithm that reads in the curve r(t), then compues the unit tangent vector, the unit normal vector, and the unit binormal to the curve. Then plot them with the curve r(t).
3. Write an algorithm that graphs the osculating plane, the normal plane and the rectifying plane along with the curve r(t).
4. Write an algorithm that graphs the curve r(t), the Frenet-Serret frame, and the osculating circle.

A good curve to test things out on is

$$r(t) = (\cos t, \sin t, t)$$

or

$$r(t) = (4\cos t, 2\sin t, t).$$

Can you make an algorithm that works in more than three dimensions?