Algebraic Geometry: additional exercises (due Nov 30)

The *projective* Grassmaniann $\mathbb{G}(M-1, N-1)$, the set of projective $(M-1)$-planes in $\mathbb{P}^{N-1}$, is isomorphic to $\mathrm{Gr}(M, N)$.

1. Consider $\mathbb{P}^3$ with the graded coordinate ring $k[x_0, x_1, x_2, x_3]$. Let $C \subset \mathbb{P}^2 \simeq \{x_3 = 0\} \subset \mathbb{P}^3$ be the plane conic
$$C = \mathbb{V}(x_3, x_0x_2 - x_1^2).$$
Find defining equations of the variety $\mathcal{C}_1(C) \subset \mathbb{G}(1, 3)$ of incident (projective) lines. (Hint: $\mathcal{C}_1(C) = \pi_1(\pi_2^{-1}(C))$, where $\pi_1 : \Sigma \to \mathbb{G}(1, 3)$ and $\pi_2 : \Sigma \to \mathbb{P}^3$ where
$$\Sigma = \{(\ell, x) \,|\, x \in \ell\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3$$
is the incidence variety.)

2. Do the same for the image of $t \mapsto [1, t, t^2, t^3] \in \mathbb{P}^3$ (twisted cubic).